

# GROTHENDIECK RING OF PRETRIANGULATED CATEGORIES

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*To the blessed memory of Andrej Nikolaevich Tyurin*

**ABSTRACT.** We consider the abelian group  $\mathcal{PT}$  generated by quasi-equivalence classes of pretriangulated DG categories with relations coming from semi-orthogonal decompositions of corresponding triangulated categories. We introduce an operation of "multiplication"  $\bullet$  on the collection of DG categories which makes this abelian group into a commutative ring. A few applications are considered: representability of "standard" functors between derived categories of coherent sheaves on smooth projective varieties and a construction of an interesting motivic measure.

## INTRODUCTION

This work grew out of an attempt to construct the Grothendieck ring of (equivalence classes of) triangulated categories. Namely, one considers an abelian group  $\mathcal{T}$  generated by equivalence classes of triangulated categories with relations coming from semiorthogonal decompositions:  $[\mathcal{A}] = [\mathcal{B}] + [\mathcal{C}]$  if  $\mathcal{A}$  admits a semiorthogonal decomposition with the two summands being equivalent to  $\mathcal{B}$  and  $\mathcal{C}$  respectively. We wanted to define a product  $\bullet$  of triangulated categories which would make  $\mathcal{T}$  a commutative associative ring. This answer was supposed to be known in the following situations:

- 1) If  $A$  and  $B$  are nice algebras (say, finite dimensional and of finite cohomological dimension), and  $D(A)$  and  $D(B)$  are the bounded derived categories of (finite)  $A$ - and  $B$ -modules respectively, then  $D(A) \bullet D(B) = D(A \otimes B)$ .
- 2) If  $X$  and  $Y$  are smooth projective varieties and  $D(X)$  and  $D(Y)$  are the corresponding bounded derived categories of coherent sheaves, then  $D(X) \bullet D(Y) = D(X \times Y)$ .

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After a while we came to conclusion that triangulated categories are not rigid enough (a well known problem being that taking the cone of a morphism is not a functorial operation), and convinced ourselves to work with *pretriangulated* categories. These are DG categories where the cone of a morphism *is* a functor, so they are more rigid and easier to work with than the triangulated categories. In a sense this is going back from the homotopy category to the abelian category of complexes. The *homotopy* category of a pretriangulated category is a triangulated category. So this approach gives a satisfactory solution to our original problem.

For example, if  $X$  and  $Y$  are smooth projective varieties and  $I(X)$  and  $I(Y)$  are the DG categories of bounded below injective complexes of  $\mathcal{O}_X$ - and  $\mathcal{O}_Y$ -modules respectively with bounded coherent cohomology, then our product gives

$$I(X) \bullet I(Y) = I(X \times Y).$$

We present two applications of this last formula. The first application is the representability of *standard* functors  $F : D(X) \rightarrow D(Y)$ . Namely, we prove that if an exact functor  $F$  comes from a DG functor between standard enhancements of  $D(X)$  and  $D(Y)$  respectively, then there is an object  $P \in D(X \times Y)$  which represents  $F$ : there exists an isomorphism of functors

$$F(\cdot) = \mathbb{R}q_*(p^*(\cdot) \overset{\mathbb{L}}{\otimes} P),$$

where  $X \overset{p}{\leftarrow} X \times Y \overset{q}{\rightarrow} Y$  are the projections. We also conjecture that every exact functor between  $D(X)$  and  $D(Y)$  is standard.

Our second application is the construction of an interesting *motivic measure*, i.e. a homomorphism from the Grothendieck ring of varieties to the Grothendieck ring of (quasi-equivalence classes) of pretriangulated categories, by sending a smooth projective variety  $X$  to  $I(X)$ . We show that the kernel of this homomorphism contains the element  $\mathbb{L} - 1$ , where  $\mathbb{L}$  is the class of the affine line.

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## 1. GENERATORS AND SEMIORTHOGONAL DECOMPOSITIONS OF TRIANGULATED CATEGORIES

Fix a field  $k$ . All categories and all functors that we will consider are assumed to be  $k$ -linear. Denote by  $Vect$  the category of  $k$ -vector spaces.

For a smooth projective variety  $X$  over  $k$  we denote  $D(X) = D^b(\text{coh}_X)$  the bounded derived category of coherent sheaves on  $X$ .

In this section we recall some results about triangulated categories following [Bo],[BoKa1],[BoVdB],[BN],[KeVo]. In the end we define the Grothendieck group  $\mathcal{T}$  of triangulated categories.

### 1. Generators and representability of cohomological functors.

Let  $\mathcal{E} = (E_i)_{i \in I}$  be a class of objects in a triangulated category  $\mathcal{A}$ . A *triangulated envelope* of  $\mathcal{E}$  is the smallest strictly full triangulated subcategory of  $\mathcal{A}$  which contains  $\mathcal{E}$ .

If  $\mathcal{A}$  is a triangulated category then a triangulated subcategory  $\mathcal{B} \subset \mathcal{A}$  is called *epaisse* (thick) if it is closed under isomorphisms and direct summands.

We say that  $\mathcal{E}$  *classically generates*  $\mathcal{A}$  if the smallest *epaisse triangulated* subcategory of  $\mathcal{A}$  containing  $\mathcal{E}$  (called the *epaisse envelope* of  $\mathcal{E}$  in  $\mathcal{A}$ ) is equal to  $\mathcal{A}$  itself. We say that  $\mathcal{A}$  is *finitely generated* if it is classically generated by one object.

By the *right orthogonal*  $\mathcal{E}^\perp$  in  $\mathcal{A}$  we denote the full subcategory of  $\mathcal{A}$  whose objects  $A$  have the property  $\text{Hom}(E_i[n], A) = 0$  for all  $i$  and  $n$ . Similarly, we define the left orthogonal  ${}^\perp\mathcal{E}$ . Clearly  $\mathcal{E}^\perp$  (and  ${}^\perp\mathcal{E}$ ) is an *epaisse* subcategory of  $\mathcal{A}$ . We say that  $\mathcal{E}$  *generates*  $\mathcal{A}$  if  $\mathcal{E}^\perp = 0$ . Clearly if  $\mathcal{E}$  classically generates  $\mathcal{A}$  then it generates  $\mathcal{A}$ , but the converse is false.

Denote by  $\text{add}(\mathcal{E})$  the minimal strictly full subcategory of  $\mathcal{A}$  which contains  $\mathcal{E}$  and is closed under taking finite direct sums and shifts. Also denote by  $\text{smd}(\mathcal{E})$  the minimal strictly full subcategory which contains  $\mathcal{E}$  and is closed under taking (possible) direct summands.

There exists a natural multiplication on the set of strictly full subcategories of  $\mathcal{A}$ . If  $\mathcal{C}$  and  $\mathcal{D}$  are two such subcategories, let  $\mathcal{C} \star \mathcal{D}$  be the strictly full subcategory whose objects  $S$  occur in exact triangles  $C \rightarrow S \rightarrow D$  with  $C \in \mathcal{C}$ ,  $D \in \mathcal{D}$ . This multiplication is associative in view of the octahedral axiom. If  $\mathcal{C}$  and  $\mathcal{D}$  are closed under direct sums and/or shifts, then so is  $\mathcal{C} \star \mathcal{D}$ .

Now we define a new multiplication operation on the set of strictly full subcategories which are closed under finite direct sums by the formula:

$$\mathcal{C} \diamond \mathcal{D} = \text{smd}(\mathcal{C} \star \mathcal{D}).$$

One can check that this operation is associative. Put

$$\langle \mathcal{E} \rangle_s = \text{add}(\mathcal{E}) \diamond \dots \diamond \text{add}(\mathcal{E}) \quad (s \text{ factors}).$$

$$\langle \mathcal{E} \rangle = \bigcup_s \langle \mathcal{E} \rangle_s$$

Thus  $\langle \mathcal{E} \rangle$  is the epaisse envelope of  $\mathcal{E}$  in  $\mathcal{A}$ . So  $\mathcal{E}$  classically generates  $\mathcal{A}$  if and only if  $\langle \mathcal{E} \rangle = \mathcal{A}$ .

**Definition 1.1.**  $\mathcal{E}$  *strongly generates*  $\mathcal{A}$  if  $\mathcal{A} = \langle \mathcal{E} \rangle_s$  for some  $s$ . We say that  $\mathcal{A}$  is *strongly finitely generated* if it is strongly generated by one object.

*Remark 1.2.* Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an exact functor between triangulated categories, which is surjective on isomorphism classes of objects. Assume that  $\mathcal{A}$  is classically (resp. strongly) generated by a collection  $\mathcal{E} \subset \mathcal{A}$ . Then  $\mathcal{B}$  is classically (resp. strongly) generated by the collection  $F(\mathcal{E})$ .

*Remark 1.3.* If a triangulated category  $\mathcal{A}$  has a strong generator, then any classical generator of  $\mathcal{A}$  is a strong one.

A triangulated category  $\mathcal{A}$  is called Ext-finite if  $\dim \oplus_n \operatorname{Hom}_{\mathcal{A}}(A, B[n]) < \infty$  for any two objects  $A$  and  $B$ .

**Definition 1.4.** Let  $\mathcal{A}$  be an Ext-finite triangulated category. A functor  $h : \mathcal{A} \rightarrow \operatorname{Vect}$  is called cohomological if it takes exact triangles to long exact sequences. We say that  $h$  is of finite type if for every object  $B \in \mathcal{A}$  the vector space  $\oplus_n h(B[n])$  is finite dimensional. Each object  $A \in \mathcal{A}$  gives rise to a cohomological functor  $h_A(\cdot) := \operatorname{Hom}(A, \cdot)$ . Similarly,  $h^A(\cdot) := \operatorname{Hom}(\cdot, A)$  defines a contravariant cohomological functor. Cohomological functors isomorphic to  $h_A$  or  $h^A$  are called representable. Note that representable functors are of finite type. The category  $\mathcal{A}$  is called left (resp. right) saturated if every cohomological (resp. contravariant cohomological) functor of finite type  $h : \mathcal{A} \rightarrow \operatorname{Vect}$  is representable. We call  $\mathcal{A}$  saturated if it is left and right saturated.

Recall that a category is called Karoubian, if every projector splits.

There is a simple criterion for a triangulated category to be saturated.

**Theorem 1.5** (BoVdB). *Assume that a triangulated category  $\mathcal{A}$  is Ext-finite, has a strong generator and is Karoubian. Then  $\mathcal{A}$  is right saturated.*

For a smooth algebraic variety  $X$  over  $k$  denote by  $D(X)$  the bounded derived category of coherent sheaves on  $X$ .

**Theorem 1.6** (BoVdB). *Let  $X$  be a smooth variety over  $k$ . Then  $D(X)$  is Karoubian and has a strong generator.*

**Corollary 1.7.** *Let  $X$  be a smooth projective variety over  $k$ . Then  $D(X)$  is right saturated.*

*Remark 1.8.* For a smooth projective  $X$  the category  $D(X)$  is equivalent to its opposite: the functor

$$\mathbf{D}(\cdot) = \mathbb{R}\underline{\mathrm{Hom}}(\cdot, \mathcal{O}_X)$$

is an anti-involution of the category  $D(X)$ . It follows that  $D(X)$  is also left saturated.

Let us also mention a few results which will be used later.

**Theorem 1.9** (BoVdB). *Let  $X$  be a smooth projective variety over  $k$ . If an object  $E \in D(X)$  generates  $D(X)$ , then it strongly generates  $D(X)$ .*

**Theorem 1.10** (BoVdB). *Let  $X, Y$  be smooth projective varieties, and let  $E^X \in D(X)$ ,  $E^Y \in D(Y)$  be respective generators. Then  $E^X \boxtimes E^Y$  is a generator of  $D(X \times Y)$ .*

**Theorem 1.11** (BN). *Assume that  $\mathcal{A}$  contains countable direct sums. Then the category  $\mathcal{A}$  is Karoubian.*

**2. Serre functors.** Let  $\mathcal{A}$  be an Ext-finite triangulated category. Recall [BoKa1] that a covariant auto-equivalence  $S : \mathcal{A} \rightarrow \mathcal{A}$  is called a Serre functor if there exists an isomorphism of bi-functors

$$\mathrm{Hom}(A, B) = \mathrm{Hom}(B, S(A))^*,$$

for  $A, B \in \mathcal{A}$ . If the Serre functor exists it is unique up to an isomorphism and is an exact functor.

*Example 1.12.* If  $X$  is a smooth projective variety of dimension  $n$ , the Serre functor on the category  $D(X)$  is  $S(\cdot) = S_X(\cdot) = (\cdot) \otimes \omega_X[n]$ , where  $\omega_X$  is the canonical line bundle on  $X$ .

*Remark 1.13.* Let  $F : \mathcal{A} \rightarrow \mathcal{B}$ ,  $G : \mathcal{B} \rightarrow \mathcal{A}$  be functors between two Ext-finite triangulated categories with the Serre functors  $S_{\mathcal{A}}$  and  $S_{\mathcal{B}}$  respectively. Assume that the functor  $F$  is the right adjoint to  $G$ , then the functor  $S_{\mathcal{B}}^{-1}FS_{\mathcal{A}}$  is the left adjoint to  $G$ .

**3. Semiorthogonal decompositions.** Let us recall some definitions and results from [BoKa1] and [Bo].

**Definition 1.14.** Let  $\mathcal{A}$  be a triangulated category,  $\mathcal{B} \subset \mathcal{A}$  – a strictly full triangulated subcategory. We call  $\mathcal{B}$  *right admissible* (resp. *left admissible*) if for every  $A \in \mathcal{A}$  there exists an exact triangle  $A_{\mathcal{B}} \rightarrow A \rightarrow A_{\mathcal{B}^\perp}$  (resp.  $A_{\mathcal{B}^\perp} \rightarrow A \rightarrow A_{\mathcal{B}}$ ) with  $A_{\mathcal{B}} \in \mathcal{B}$  and  $A_{\mathcal{B}^\perp} \in \mathcal{B}^\perp$  (resp.  $A_{\mathcal{B}^\perp} \in {}^\perp\mathcal{B}$ ). A subcategory is called *admissible* if it is both left and right admissible.

Clearly a strictly full triangulated subcategory  $\mathcal{B} \subset \mathcal{A}$  is right (resp. left) admissible if and only if  $\mathcal{B}^\perp$  (resp.  ${}^\perp\mathcal{B}$ ) is left (resp. right) admissible.

**Lemma 1.15.** *Let  $\mathcal{B} \subset \mathcal{A}$  be right (resp. left) admissible. Then  ${}^\perp(\mathcal{B}^\perp) = \mathcal{B}$  (resp.  $({}^\perp\mathcal{B})^\perp = \mathcal{B}$ ).*

The next proposition appeared in [KeVo],[Bo],[BoKa].

**Proposition 1.16.** *Let  $\mathcal{A}$  be a triangulated category,  $\mathcal{B} \subset \mathcal{A}$  – a strictly full triangulated subcategory. The following conditions are equivalent:*

- a)  $\mathcal{B}$  is right (resp. left) admissible in  $\mathcal{A}$ ;
- b) the embedding functor  $i : \mathcal{B} \rightarrow \mathcal{A}$  has a right (resp. left) adjoint  $i^!$  (resp.  $i^*$ ).

*If these hold, then the compositions  $i^! \cdot i$  and  $i^* \cdot i$  are isomorphic to the identity functor on  $\mathcal{B}$ .*

**Proof.** (A sketch. See [KeVo] or [Bo] for details). If  $\mathcal{B}$  is right admissible then for each  $A \in \mathcal{A}$  the triangle  $A_{\mathcal{B}} \rightarrow A \rightarrow A_{\mathcal{B}^\perp}$  is unique up to an isomorphism. Moreover, the correspondence  $A \mapsto A_{\mathcal{B}}$  extends to an exact functor from  $\mathcal{A}$  to  $\mathcal{B}$ . This functor is  $i^!$ . Conversely, given a left adjoint  $i^!$  to  $i$  for any  $A \in \mathcal{A}$  consider the adjunction morphism  $\alpha_A : i \cdot i^! A \rightarrow A$ . The required exact triangle is

$$i \cdot i^! A \rightarrow A \rightarrow C(\alpha_A).$$

Similarly for left admissible  $\mathcal{B}$ . □

**Corollary 1.17.** *a) Let  $\mathcal{C}$  be a right admissible subcategory of  $\mathcal{B}$ , which is a right admissible subcategory of  $\mathcal{A}$ . Then  $\mathcal{C}$  is right admissible in  $\mathcal{A}$ . Similarly for left admissible subcategories. b) Let  $\mathcal{C}$  be a right admissible subcategory of  $\mathcal{A}$ . Assume that  $\mathcal{B}$  is a full subcategory of  $\mathcal{A}$  which contains  $\mathcal{C}$ . Then  $\mathcal{C}$  is right admissible in  $\mathcal{B}$ .*

**Proof.** a) Indeed, let  $i : \mathcal{C} \rightarrow \mathcal{B}$ ,  $j : \mathcal{B} \rightarrow \mathcal{A}$  be the embeddings and  $i^!$ ,  $j^!$  be the corresponding right adjoints. Then  $(j \cdot i)^! = i^! \cdot j^!$ . b) Indeed, the restriction of the adjoint functor of the embedding functor  $\mathcal{C} \rightarrow \mathcal{A}$  to the subcategory  $\mathcal{B}$  is the adjoint to the embedding  $\mathcal{C} \rightarrow \mathcal{B}$ . □

*Remark 1.18.* Let  $i : \mathcal{B} \rightarrow \mathcal{A}$  be an inclusion of a right (resp. left) admissible subcategory. Assume that  $\mathcal{E} \subset \mathcal{A}$  (strongly) generates  $\mathcal{A}$ . Then  $i^!(\mathcal{E})$  (resp.  $i^*(\mathcal{E})$ ) (strongly) generates  $\mathcal{B}$ . Indeed, this follows from Remark 1.2 since the functors  $i^!$  and  $i^*$  are surjective.

*Remark 1.19.* A right or left admissible subcategory  $\mathcal{B} \subset \mathcal{A}$  is épaisse. Hence if  $\mathcal{A}$  is Karoubian, then  $\mathcal{B}$  is also such.

**Lemma 1.20.** *Let  $\mathcal{A}$  be a triangulated category,  $\mathcal{B} \subset \mathcal{A}$  – an epaisse triangulated subcategory. Put  $\mathcal{C} = \mathcal{B}^\perp$ . Let  $\mathcal{A}$  be classically generated by a collection of objects  $\mathcal{E} \subset \mathcal{A}$ . Assume that for each  $E \in \mathcal{E}$  there is an exact triangle*

$$E_{\mathcal{B}} \rightarrow E \rightarrow E_{\mathcal{C}},$$

*where  $E_{\mathcal{B}} \in \mathcal{B}$  and  $E_{\mathcal{C}} \in \mathcal{C}$ . Then the subcategory  $\mathcal{B}$  is right admissible in  $\mathcal{A}$ . Similarly for left admissible categories. Moreover, if  $\mathcal{D} \subset \mathcal{C}$  is the strictly full subcategory consisting of all objects  $A_{\mathcal{C}}$  for  $A \in \mathcal{A}$ , then  $\mathcal{D} = \mathcal{C}$ .*

**Proof.** The proof is similar to the proof of Proposition 1.16 above. Namely, the association  $E \mapsto E_{\mathcal{B}}$  can be extended to a functor from the full subcategory, whose objects is the collection  $\mathcal{E}$ , to  $\mathcal{B}$ . This functor extends to an exact functor from the triangulated envelope of  $\mathcal{E}$  to  $\mathcal{B}$ . Finally, since  $\mathcal{B}$  is epaisse, this functor extends to the epaisse envelope of  $\mathcal{E}$ , i.e. to  $\mathcal{A}$ . This functor  $\mathcal{A} \rightarrow \mathcal{B}$  is the right adjoint to the inclusion functor  $\mathcal{B} \rightarrow \mathcal{A}$ . So  $\mathcal{B}$  is right admissible in  $\mathcal{A}$  by Proposition 1.16

Let us prove the last assertion of the lemma. Let  $C \in \mathcal{C}$  and consider the canonical triangle

$$C_{\mathcal{B}} \rightarrow C \rightarrow C_{\mathcal{C}}.$$

The map  $C_{\mathcal{B}} \rightarrow C$  is zero, hence  $C_{\mathcal{C}}$  is isomorphic to  $C \oplus C_{\mathcal{B}}[1]$ , which implies that  $C_{\mathcal{B}} = 0$ . Hence  $C \in \mathcal{D}$ .  $\square$

The following two statements establish a relation between saturated and admissible categories.

**Lemma 1.21** (BoKa1). *Let  $\mathcal{A}$  be a right (resp. left) saturated triangulated category, and  $\mathcal{B} \subset \mathcal{A}$  be a left (resp. right) admissible subcategory. Then  $\mathcal{B}$  is right (resp. left) saturated.*

**Proposition 1.22** (BoKa1). *Let  $\mathcal{B}$  be a strictly full triangulated subcategory of a triangulated category  $\mathcal{A}$  of finite type. Assume that  $\mathcal{B}$  is right (resp. left) saturated. Then  $\mathcal{B}$  is right (resp. left) admissible in  $\mathcal{A}$ .*

**Definition 1.23.** Let  $\mathcal{A}$  be a triangulated category. We say that admissible subcategories  $\mathcal{B}_1, \dots, \mathcal{B}_n \subset \mathcal{A}$  form a *semiorthogonal decomposition* of  $\mathcal{A}$ , denoted  $\mathcal{A} = (\mathcal{B}_1, \dots, \mathcal{B}_n)$  if  $\mathcal{A}$  is the triangulated envelope of  $\cup_i \text{Ob} \mathcal{B}_i$ , and  $\mathcal{B}_j \subset {}^\perp \mathcal{B}_i$  for  $j > i$ . We call  $\mathcal{B}_i$ 's *semiorthogonal summands* of  $\mathcal{A}$ .

*Example 1.24.* Let  $\mathcal{B}$  be an admissible subcategory of a triangulated category  $\mathcal{A}$ . Assume that  $\mathcal{B}^\perp$  is right admissible. Then  $\mathcal{A} = (\mathcal{B}, \mathcal{B}^\perp)$  is a semiorthogonal decomposition.

**Lemma 1.25.** *Let  $\mathcal{A} = (\mathcal{B}, \mathcal{C})$  be a semiorthogonal decomposition. Then  $\mathcal{B} = \mathcal{C}^\perp$ .*

**Proof.** Let  $\mathcal{D} \subset \mathcal{C}^\perp$  be the right orthogonal to  $\mathcal{B}$  in  $\mathcal{C}^\perp$ . Then  $\mathcal{D} = 0$ , because  $\mathcal{A}$  is the triangulated envelope of  $\mathcal{B} \cup \mathcal{C}$ . By Corollary 1.17 above the category  $\mathcal{B}$  is admissible in  $\mathcal{C}^\perp$  and hence  $\mathcal{B} = \mathcal{C}^\perp$ .  $\square$

*Remark 1.26.* Let  $\mathcal{A} = (\mathcal{B}_1, \dots, \mathcal{B}_n)$  be a semiorthogonal decomposition. Assume that for each  $i$  there is given a semiorthogonal decomposition  $\mathcal{B}_i = (\mathcal{C}_i^1, \dots, \mathcal{C}_i^{k_i})$ . Then by Corollary 1.17 each  $\mathcal{C}_i^j$  is admissible in  $\mathcal{A}$  and  $\mathcal{A} = (\mathcal{C}_1^1, \dots, \mathcal{C}_n^{k_n})$  is a semiorthogonal decomposition.

**Definition 1.27.** Let  $\mathcal{T}$  be the abelian group generated by equivalence classes of triangulated categories with the relations coming from semiorthogonal decompositions. Namely, we put  $[\mathcal{A}] = [\mathcal{B}] + [\mathcal{C}]$  if there exists a semiorthogonal decomposition  $\mathcal{A}_1 = (\mathcal{B}_1, \mathcal{B}_2)$  with  $\mathcal{A}_1, \mathcal{B}_1, \mathcal{B}_2$  being equivalent to  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  respectively. We call  $\mathcal{T}$  the *Grothendieck group* of triangulated categories.

This work grew out of an attempt to make  $\mathcal{T}$  into a commutative ring by defining an appropriate product of triangulated categories.

## 2. TENSOR PRODUCT OF TRIANGULATED CATEGORIES: A FAILED ATTEMPT

Recall that there is a simple construction of the tensor product of preadditive (or prelinear) categories. Namely, if  $\mathcal{E}$  and  $\mathcal{F}$  are two such categories, we let the objects of  $\mathcal{E} \otimes \mathcal{F}$  to be the pairs  $(X, Y) \in \text{Ob}\mathcal{E} \times \text{Ob}\mathcal{F}$  with  $\text{Mor}((X_1, Y_1), (X_2, Y_2)) = \text{Mor}(X_1, X_2) \otimes \text{Mor}(Y_1, Y_2)$ .

An axiomatic definition of the tensor product of abelian categories was given by Deligne in [De]. Let us recall it. Let  $\{\mathcal{A}_i\}_{i \in I}$  be a collection of ( $k$ -linear) abelian categories. An abelian category  $\mathcal{A}$  together with a functor

$$\otimes : \prod \mathcal{A}_i \longrightarrow \mathcal{A}$$

which is right exact in each variable is called the *tensor product* of  $\mathcal{A}_i$ 's if it has the following universal property: For any abelian category  $\mathcal{C}$  the composition with the functor  $\otimes$  establishes an equivalence of categories of right exact functors  $\mathcal{A} \rightarrow \mathcal{C}$  and functors  $\prod \mathcal{A}_i \rightarrow \mathcal{C}$ , which are right exact in each variable.

Deligne proves the existence of the tensor product under some finiteness condition on the categories  $\mathcal{A}_i$ . Namely, every object should have finite length, and the Hom-space between any two objects should be finite dimensional. Actually in this case the functor  $\otimes$  is exact in each variable.



One could try to give a similar definition for triangulated categories, say by replacing right exact functors by exact ones (in the triangulated sense). However this approach meets difficulties when one tries to prove the existence, because the triangulated categories are not as rigid as abelian ones (since taking the cone of a morphism is not a functorial operation). Also one might want the tensor product of saturated triangulated categories to be also a saturated triangulated category. In this case a good candidate seems to be  $Fun_{multi-ex}(\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n, D^b(Vect))$  – the category of multi-exact functors from  $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n$  to  $D^b(Vect)$ . Unfortunately, this last category does not have a visible triangulated structure.

So we came to conclusion that it is necessary to consider *enhanced* triangulated categories.

### 3. DG CATEGORIES, PRETRIANGULATED CATEGORIES, ENHANCED TRIANGULATED CATEGORIES

Our main references here are [BoKa2],[Dr],[Ke].

**1. DG categories.** A DG category is an additive category  $\mathcal{A}$  in which the sets  $\text{Hom}(A, B)$ ,  $A, B \in \text{Ob}\mathcal{A}$ , are provided with a structure of a  $\mathbb{Z}$ -graded  $k$ -module and a differential  $d : \text{Hom}(A, B) \rightarrow \text{Hom}(A, B)$  of degree 1, so that for every  $A, B, C \in \mathcal{A}$  the composition  $\text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$  comes from a morphism of complexes  $\text{Hom}(A, B) \otimes \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$ . Also there is a closed degree zero morphism  $1_A \in \text{Hom}(A, A)$  which behaves as the identity under composition of morphisms.

The simplest example of a DG category is the category  $DG(k)$  of complexes of  $k$ -vector spaces, or DG  $k$ -modules.

Using the supercommutativity isomorphism  $S \otimes T \simeq T \otimes S$  in the category of DG  $k$ -modules one defines for every DG category  $\mathcal{A}$  the opposite DG category  $\mathcal{A}^0$  with  $\text{Ob}\mathcal{A}^0 = \text{Ob}\mathcal{A}$ ,  $\text{Hom}_{\mathcal{A}^0}(A, B) = \text{Hom}_{\mathcal{A}}(B, A)$ . We denote by  $\mathcal{A}^{gr}$  the *graded* category which is obtained from  $\mathcal{A}$  by forgetting the differentials on  $\text{Hom}$ 's.

The tensor product of DG-categories  $\mathcal{A}$  and  $\mathcal{B}$  is defined as follows:

(i)  $\text{Ob}(\mathcal{A} \otimes \mathcal{B}) := \text{Ob}\mathcal{A} \times \text{Ob}\mathcal{B}$ ; for  $A \in \text{Ob}\mathcal{A}$  and  $B \in \text{Ob}\mathcal{B}$  the corresponding object is denoted by  $A \otimes B$ ;

(ii)  $\text{Hom}(A \otimes B, A' \otimes B') := \text{Hom}(A, A') \otimes \text{Hom}(B, B')$  and the composition map is defined by  $(f_1 \otimes g_1)(f_2 \otimes g_2) := (-1)^{\deg(g_1) \deg(f_2)} f_1 f_2 \otimes g_1 g_2$ .

Note that the DG categories  $\mathcal{A} \otimes \mathcal{B}$  and  $\mathcal{B} \otimes \mathcal{A}$  are canonically isomorphic. In the above notation the isomorphism functor  $\phi$  is

$$\phi(A \otimes B) = (B \otimes A), \quad \phi(f \otimes g) = (-1)^{\deg(f) \deg(g)} (g \otimes f).$$

Given a DG-category  $\mathcal{A}$  one defines the graded category  $Ho(\mathcal{A})$  with  $Ob Ho(\mathcal{A}) = Ob \mathcal{A}$  by replacing each Hom complex by the direct sum of its cohomology groups. We call  $Ho(\mathcal{A})$  the *graded homotopy category* of  $\mathcal{A}$ . Restricting ourselves to the 0-th cohomology of the Hom complexes we get the *homotopy category*  $Ho(\mathcal{A})$ .

Two objects  $A, B \in Ob \mathcal{A}$  are called DG *isomorphic* (or, simply, isomorphic) if there exists an invertible degree zero morphism  $f \in Hom(A, B)$ . We say that  $A, B$  are *homotopy equivalent* if they are isomorphic in  $Ho(\mathcal{A})$ .

A DG-functor between DG-categories  $F : \mathcal{A} \rightarrow \mathcal{B}$  is said to be a *quasi-equivalence* if  $Ho(F) : Ho(\mathcal{A}) \rightarrow Ho(\mathcal{B})$  is full and faithful and  $Ho(F)$  is essentially surjective. We say that  $F$  is a DG *equivalence* if it is full and faithful and every object of  $\mathcal{B}$  is DG isomorphic to an object of  $F(\mathcal{A})$ . Certainly, a DG equivalence is a quasi-equivalence. DG categories  $\mathcal{C}$  and  $\mathcal{D}$  are called *quasi-equivalent* if there exist DG categories  $\mathcal{A}_1, \dots, \mathcal{A}_n$  and a chain of quasi-equivalences

$$\mathcal{C} \leftarrow \mathcal{A}_1 \rightarrow \dots \leftarrow \mathcal{A}_n \rightarrow \mathcal{D}.$$

Given DG categories  $\mathcal{A}$  and  $\mathcal{B}$  the collection of covariant DG functors  $\mathcal{A} \rightarrow \mathcal{B}$  is itself the collection of objects of a DG category, which we denote by  $Fun_{DG}(\mathcal{A}, \mathcal{B})$ . Namely, let  $\phi$  and  $\psi$  be two DG functors. Put  $Hom^k(\phi, \psi)$  equal to the set of natural transformations  $t : \phi^{gr} \rightarrow \psi^{gr}[k]$  of graded functors from  $\mathcal{A}^{gr}$  to  $\mathcal{B}^{gr}$ . This means that for any morphism  $f \in Hom_{\mathcal{A}}^s(A, B)$  one has

$$\psi(f) \cdot t(A) = (-1)^{ks} t(B) \cdot \phi(f).$$

On each  $A \in \mathcal{A}$  the differential of the transformation  $t$  is equal to  $(dt)(A)$  (one easily checks that this is well defined). Thus, the closed transformations of degree 0 are the DG transformations of DG functors. A similar definition gives us the DG-category consisting of the contravariant DG functors  $Fun_{DG}(\mathcal{A}^0, \mathcal{B}) = Fun_{DG}(\mathcal{A}, \mathcal{B}^0)$  from  $\mathcal{A}$  to  $\mathcal{B}$ .

*Remark 3.1.* Let  $\mathcal{A}$  and  $\mathcal{B}$  be DG categories. Note that the category  $Ho(\mathcal{A}) \otimes Ho(\mathcal{B})$  (and hence  $Ho(\mathcal{A}) \times Ho(\mathcal{B})$ ) is a (not full, in general) subcategory of  $Ho(\mathcal{A} \otimes \mathcal{B})$ .

**2. DG modules over DG categories.** We denote the DG category  $Fun_{DG}(\mathcal{B}, DG(k))$  by  $\mathcal{B}\text{-mod}$  and call it the category of DG  $\mathcal{B}$ -modules. There is a natural covariant DG functor  $h : \mathcal{A} \rightarrow \mathcal{A}^0\text{-mod}$  (the Yoneda embedding) defined by  $h^A(B) := Hom_{\mathcal{A}}(B, A)$ . As in the "classical" case one verifies that the functor  $h$  is full and faithful, i.e.

$$Hom_{\mathcal{A}}(A, A') = Hom_{\mathcal{A}^0\text{-mod}}(h^A, h^{A'}).$$

Moreover, for any  $F \in \mathcal{A}^0\text{-mod}$ ,  $A \in \mathcal{A}$

$$\text{Hom}_{\mathcal{A}^0\text{-mod}}(h^A, F) = F(A).$$

The  $\mathcal{A}^0$ -DG-modules  $h^A$ ,  $A \in \mathcal{A}$  are called *free*. An  $\mathcal{A}^0$ -DG-module  $F$  is called *semi-free* if it has a filtration

$$0 = F_0 \subset F_1 \subset \dots = F,$$

such that  $F_{i+1}/F_i$  is isomorphic to a direct sum of shifted free  $\mathcal{A}^0$ -DG-modules  $h^A[n]$ ,  $n \in \mathbb{Z}$ . The full subcategory of semi-free  $\mathcal{A}^0$ -DG-modules is denoted by  $SF(\mathcal{A})$ .

An  $\mathcal{A}^0$ -DG-module  $F$  is called *acyclic*, if the complex  $F(A)$  is acyclic for all  $A \in \mathcal{A}$ . Let  $D(\mathcal{A})$  denote the *derived category* of  $\mathcal{A}^0$ -DG-modules, i.e.  $D(\mathcal{A})$  is the Verdier quotient of the homotopy category  $Ho(\mathcal{A}^0\text{-mod})$  by the subcategory of acyclic DG-modules. The following proposition was essentially proved in [Ke].

**Proposition 3.2** (Dr). *The inclusion functor  $SF(\mathcal{A}) \hookrightarrow \mathcal{A}^0\text{-mod}$  induces an equivalence of triangulated categories  $Ho(SF(\mathcal{A})) \simeq D(\mathcal{A})$ .*

A DG functor  $G : \mathcal{A} \rightarrow \mathcal{B}$  induces DG functors

$$\text{Res}_G : \mathcal{B}^0\text{-mod} \rightarrow \mathcal{A}^0\text{-mod}, \quad \text{Ind}_G : \mathcal{A}^0\text{-mod} \rightarrow \mathcal{B}^0\text{-mod},$$

where  $\text{Res}_G$  is the obvious restriction functor. We refer to [Dr] for the definition of  $\text{Ind}_G$ ; let us only list some of the properties.

1). The functor  $\text{Ind}_G$  is left adjoint to  $\text{Res}_G$ , that is for every  $\Phi \in \mathcal{A}^0\text{-mod}$ ,  $\Psi \in \mathcal{B}^0\text{-mod}$  there is a canonical isomorphism of complexes

$$\text{Hom}_{\mathcal{B}^0\text{-mod}}(\text{Ind}_G(\Phi), \Psi) = \text{Hom}_{\mathcal{A}^0\text{-mod}}(\Phi, \text{Res}_G(\Psi)).$$

2). For any  $A \in \mathcal{A}$ ,  $\text{Ind}_G(h^A) = h^{(G(A))}$ , which means that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{h} & \mathcal{A}^0\text{-mod} \\ G \downarrow & & \downarrow \text{Ind}_G \\ \mathcal{B} & \xrightarrow{h} & \mathcal{B}^0\text{-mod}, \end{array}$$

where the horizontal arrows are the Yoneda embeddings.

3). The functor  $\text{Ind}_G$  preserves semi-free DG modules and  $\text{Ind}_G : SF(\mathcal{A}) \rightarrow SF(\mathcal{B})$  is a quasi-equivalence if  $G$  is such.

**Corollary 3.3** (Ke). *If DG categories  $\mathcal{A}$  and  $\mathcal{B}$  are quasi-equivalent, then the derived categories  $D(\mathcal{A})$  and  $D(\mathcal{B})$  are equivalent.*

**Proof.** Indeed, this follows from Proposition 3.2 and the last property of the functor  $\text{Ind}_G$ .  $\square$

It is natural to consider the category of  $h$ -projective  $\mathcal{A}^0$ -DG-modules which we introduce next. Namely, we call a  $\mathcal{A}^0$ -DG-module  $P$  *h-projective* if

$$\text{Hom}_{\text{Ho}(\mathcal{A}^0\text{-mod})}(P, F) = 0$$

for every acyclic  $F \in \mathcal{A}^0\text{-mod}$ . Let  $\mathcal{P}(\mathcal{A}) \subset \mathcal{A}^0\text{-mod}$  denote the full subcategory of  $h$ -projective objects. It can be shown that a semi-free  $\mathcal{A}^0$ -DG-module is  $h$ -projective. Hence the Proposition 3.2 implies the equivalences

$$\text{Ho}(SF(\mathcal{A})) \simeq \text{Ho}(\mathcal{P}(\mathcal{A})) \simeq D(\mathcal{A}).$$

Property 3 of the functor  $\text{Ind}_G$  implies the following corollary.

**Corollary 3.4.** *If DG categories  $\mathcal{A}$  and  $\mathcal{B}$  are quasi-equivalent, then also  $\mathcal{P}(\mathcal{A})$  and  $\mathcal{P}(\mathcal{B})$  are quasi-equivalent.*

**3. Pretriangulated DG categories.** Given a DG-category  $\mathcal{A}$  one can associate to it a triangulated category  $\mathcal{A}^{tr}$  [BoKa2]. It is defined as the homotopy category of a certain DG-category  $\mathcal{A}^{\text{pre-tr}}$ . The idea of the definition of  $\mathcal{A}^{\text{pre-tr}}$  is to formally add cones of all morphisms, cones of morphisms between cones, etc.

First we need to clarify the notion of a "formal shift" of an object.

**Definition 3.5.** We define the DG category  $\bar{\mathcal{A}}$  as follows:

$$\text{Ob}\bar{\mathcal{A}} = \{A[n] \mid A \in \text{Ob}\mathcal{A}, n \in \mathbb{Z}\},$$

and

$$\text{Hom}_{\bar{\mathcal{A}}}(A[k], B[n]) = \text{Hom}_{\mathcal{A}}(A, B)[n - k]$$

as *graded vector spaces*. If  $f \in \text{Hom}_{\mathcal{A}}(A, B)$  is considered as an element of  $\text{Hom}_{\bar{\mathcal{A}}}(A[k], B[n])$  under the above identification then the differentials are related by the formula

$$d_{\bar{\mathcal{A}}}(f) = (-1)^n d_{\mathcal{A}}(f).$$

Notice, for example, that the differential in  $\text{Hom}_{\bar{\mathcal{A}}}(A[1], B[1])$  is equal to *minus* the differential in  $\text{Hom}_{\mathcal{A}}(A, B)$ .

One can check that the composition of morphisms in  $\bar{\mathcal{A}}$  is compatible with the Leibniz rule, so that  $\bar{\mathcal{A}}$  is indeed a DG category. Clearly  $\bar{\mathcal{A}}$  contains  $\mathcal{A}$  as a full DG subcategory.

Given an object  $A \in \mathcal{A}$  the object  $A[r]$  is characterized (up to a DG isomorphism) by the existence of closed morphisms  $f : A \rightarrow A[r]$ ,  $g : A[r] \rightarrow A$  of degrees  $-r$  and  $r$  respectively, such that  $fg = gf = 1$ . Thus in particular every DG functor commutes with shifts.

**Definition 3.6.** The objects of  $\mathcal{A}^{\text{pre-tr}}$  are "one-sided twisted complexes", i.e. formal expressions  $(\oplus_{i=1}^n C_i[r_i], q)$ , where  $C_i \in \text{Ob}\mathcal{A}$ ,  $r_i \in \mathbb{Z}$ ,  $n \geq 0$ ,  $q = (q_{ij})$ ,  $q_{ij} \in \text{Hom}(C_j[r_j], C_i[r_i])$  is homogeneous of degree 1,  $q_{ij} = 0$  for  $i \geq j$ ,  $dq + q^2 = 0$ . If  $C, C' \in \text{Ob}\mathcal{A}^{\text{pre-tr}}$ ,  $C = (\oplus C_j[r_j], q)$ ,  $C' = (\oplus C'_j[r'_j], q')$ , the  $\mathbb{Z}$ -graded  $k$ -module  $\text{Hom}(C, C')$  is the space of matrices  $f = (f_{ij})$ ,  $f_{ij} \in \text{Hom}(C_j[r_j], C'_i[r'_i])$ , and the composition map  $\text{Hom}(C, C') \otimes \text{Hom}(C', C'') \rightarrow \text{Hom}(C, C'')$  is matrix multiplication. The differential  $d : \text{Hom}(C, C') \rightarrow \text{Hom}(C, C')$  is defined by  $df := (df_{ij}) + q'f - (-1)^l f q$  if  $\deg f_{ij} = l$ .

Notice that the DG category  $\mathcal{A}^{\text{pre-tr}}$  is closed under formal shifts:

$$(\oplus_{i=1}^n C_i[r_i], q)[1] = (\oplus_{i=1}^n C_i[r_i + 1], -q).$$

**Definition 3.7.** Let  $\mathcal{B}$  be a DG category and  $f \in \text{Hom}(A, B)$  be a closed degree zero morphism in  $\mathcal{B}$ . An object  $C \in \mathcal{B}$  is called the cone of  $f$ , denoted  $\text{Cone}(f)$ , if  $\mathcal{B}$  contains the object  $A[1]$  and there exist degree zero morphisms

$$A[1] \xrightarrow{i} C \xrightarrow{p} A[1], \quad B \xrightarrow{j} C \xrightarrow{s} B$$

with the properties

$$pi = 1, \quad sj = 1, \quad si = 0, \quad pj = 0, \quad ip + js = 1,$$

and

$$d(j) = d(p) = 0, \quad d(i) = jf, \quad d(s) = -fp.$$

**Lemma 3.8.** *The cone of a closed degree zero morphism is uniquely defined up to a DG isomorphism.*

**Proof.** Note that the first set of conditions means that  $C$  is the direct sum of  $A[1]$  and  $B$  in the corresponding graded category  $\mathcal{B}^{\text{gr}}$ . Thus for any object  $E$  in  $\mathcal{A}$  there are isomorphisms of graded  $k$ -modules

$$\text{Hom}(E, C) = \text{Hom}(E, A[1]) \oplus \text{Hom}(E, B),$$

$$\text{Hom}(C, E) = \text{Hom}(A[1], E) \oplus \text{Hom}(B, E),$$

which are given by composing with  $i$  and  $j$  (or with  $p$  and  $s$ ). Then the second set of conditions determines the differentials in  $\text{Hom}(E, C)$  and  $\text{Hom}(C, E)$ .  $\square$

Given a closed degree zero morphism  $f : A \rightarrow B$  the diagram

$$A \xrightarrow{f} B \xrightarrow{j} \text{Cone}(f) \xrightarrow{p} A[1]$$

is called a preexact triangle.

*Remark 3.9.* It is clear that any DG functor preserves cones of closed degree zero morphisms and preserves preexact triangles.

**Proposition 3.10** (BoKa2). *Let  $\mathcal{A}$  be a DG category. Then*

- a) *the DG category  $\mathcal{A}^{\text{pre-tr}}$  is closed under taking cones of closed degree zero morphisms;*
- b) *every object in  $\mathcal{A}^{\text{pre-tr}}$  can be obtained from objects in  $\mathcal{A}$  by taking successive cones of closed degree zero morphisms.*

**Proof.** a). Given a closed morphism of degree zero

$$f : (\oplus C_i[r_i], q) \rightarrow (\oplus C'_j[r'_j], q')$$

its cone is the twisted complex  $(\oplus C'_j[r'_j] \oplus C_i[r_i + 1], (q', -q + f))$ . For example, if  $A, B \in \mathcal{A}$  and  $f : A \rightarrow B$  is a closed morphism of degree 0 then  $\text{Cone}(f)$  is the twisted complex  $(B \oplus A[1], (0, f)) \in \mathcal{A}^{\text{pre-tr}}$ .

b). Let  $C = (\oplus_{i=1}^n C_i[r_i], q)$  be a twisted complex. Consider its twisted subcomplex  $C' = (\oplus_{i=1}^{n-1} C_i[r_i], q')$ , where  $q' = q - \oplus_i q_{in}$ . Then  $C$  is the cone of the closed degree zero morphism  $\oplus_{i=1}^{n-1} q_{in} : (C_n[r_n - 1], 0) \rightarrow C'$ .  $\square$

A DG category  $\mathcal{A}$  is said to be *pretriangulated* if for every  $A \in \mathcal{A}$ ,  $k \in \mathbb{Z}$  the object  $A[k] \in \mathcal{A}^{\text{pre-tr}}$  is homotopy equivalent to an object of  $\mathcal{A}$  and for every closed morphism of degree zero  $f$  in  $\mathcal{A}$  the object  $\text{Cone}(f) \in \mathcal{A}^{\text{pre-tr}}$  is homotopy equivalent to an object of  $\mathcal{A}$ . We say that  $\mathcal{A}$  is *strongly pretriangulated* if the same is true with "homotopy equivalent" replaced by "DG isomorphic". Actually, if  $\mathcal{A}$  is pretriangulated (resp. strongly pretriangulated) then every object of  $\mathcal{A}^{\text{pre-tr}}$  is homotopy equivalent (resp. DG isomorphic) to an object of  $\mathcal{A}$  [Dr]. Thus  $\mathcal{A}$  is pretriangulated (resp. strongly pretriangulated) if and only if the embedding  $\text{Ho}(\mathcal{A}) \hookrightarrow \text{Ho}(\mathcal{A}^{\text{pre-tr}})$  is an equivalence (resp. the embedding  $\mathcal{A} \hookrightarrow \mathcal{A}^{\text{pre-tr}}$  is a DG equivalence).

**Proposition 3.11.** *Let  $\mathcal{A}, \mathcal{B}$  be DG categories and  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a DG functor. Assume that  $\mathcal{B}$  is strongly pretriangulated. Then there exists a unique (up to a DG isomorphism) DG functor  $G : \mathcal{A}^{\text{pre-tr}} \rightarrow \mathcal{B}$  which extends  $F$ .*

**Proof.** The canonical embedding  $J : \mathcal{B} \rightarrow \mathcal{B}^{\text{pre-tr}}$  is a DG equivalence. Thus we may put  $G = J^{-1} \cdot F^{\text{pre-tr}}$ . The uniqueness of  $G$  follows again from part b) of Proposition 3.10.  $\square$

For a DG-category  $\mathcal{A}$  the DG categories  $\mathcal{A}^{\text{pre-tr}}$ ,  $\mathcal{A}^0\text{-mod}$ ,  $SF(\mathcal{A})$ ,  $\mathcal{P}(\mathcal{A})$  are strongly pretriangulated. Moreover, if  $\mathcal{A}$  is a DG category and  $\mathcal{B}$  is a strongly pretriangulated category, then the DG category  $\text{Fun}_{DG}(\mathcal{A}, \mathcal{B})$  is strongly pretriangulated.

If  $\mathcal{A}$  is pretriangulated then every closed degree 0 morphism  $f : A \rightarrow B$  in  $\mathcal{A}$  gives rise to the usual triangle  $A \rightarrow B \rightarrow \text{Cone}(f) \rightarrow A[1]$  in  $\text{Ho}(\mathcal{A})$ . Triangles of this type and those isomorphic to them are called

*exact.* If  $\mathcal{A}$  is pretriangulated then  $Ho(\mathcal{A})$  becomes a triangulated category. So  $\mathcal{A}^{tr} := Ho(\mathcal{A}^{pre-tr})$  is a triangulated category.

Let  $F : \mathcal{A} \hookrightarrow \mathcal{A}^{pre-tr}$  denote (temporarily) the canonical (full and faithful) embedding. We have the commutative diagrams

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{h_{\mathcal{A}}} & \mathcal{A}^0\text{-mod} \\ F \downarrow & & \downarrow \text{Ind}_F \\ \mathcal{A}^{pre-tr} & \xrightarrow{h_{\mathcal{A}^{pre-tr}}} & (\mathcal{A}^{pre-tr})^0\text{-mod} \end{array}$$

and

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{h_{\mathcal{A}}} & \mathcal{A}^0\text{-mod} \\ F \downarrow & & \uparrow \text{Res}_F \\ \mathcal{A}^{pre-tr} & \xrightarrow{h_{\mathcal{A}^{pre-tr}}} & (\mathcal{A}^{pre-tr})^0\text{-mod} \end{array}$$

The DG functors  $\text{Ind}_F$  and  $\text{Res}_F$  are mutually inverse DG equivalences, which preserve the subcategories of semi-free DG modules. We denote the full and faithful DG functor

$$\alpha = \alpha_{\mathcal{A}} := \text{Res}_F \cdot h_{\mathcal{A}^{pre-tr}} : \mathcal{A}^{pre-tr} \rightarrow \mathcal{A}^0\text{-mod}.$$

Then  $\alpha(\mathcal{A}^{pre-tr})$  is a strictly full subcategory of  $SF(\mathcal{A})$ .

*Remark 3.12 (Dr).* If a DG functor  $G : \mathcal{A} \rightarrow \mathcal{B}$  is a quasi-equivalence, then the corresponding quasi-equivalence  $\text{Ind}_G : SF(\mathcal{A}) \rightarrow SF(\mathcal{B})$  induces a quasi-equivalence

$$\text{Ind}_G : \alpha(\mathcal{A}^{pre-tr}) \rightarrow \alpha(\mathcal{B}^{pre-tr}).$$

#### 4. Perfect DG modules.

**Definition 3.13.** Let  $\mathcal{A}$  be a DG category. Consider the full pretriangulated subcategory  $\alpha(\mathcal{A}^{pre-tr}) \subset SF(\mathcal{A})$ . Let  $\text{Perf-}\mathcal{A}$  be the full DG subcategory of  $SF(\mathcal{A})$  consisting of DG modules which are homotopy equivalent to a direct summand of an object in  $\alpha(\mathcal{A}^{pre-tr})$ . We call these the *perfect*  $\mathcal{A}^0$ -DG-modules.

The full subcategory  $Ho(\text{Perf-}\mathcal{A}) \subset Ho(SF(\mathcal{A}))$  is the epaisse envelope of the triangulated subcategory  $Ho(\alpha(\mathcal{A}^{pre-tr})) \subset Ho(SF(\mathcal{A}))$ . Hence  $Ho(\text{Perf-}\mathcal{A})$  is also triangulated and, therefore,  $\text{Perf-}\mathcal{A}$  is a (strongly) pretriangulated subcategory of  $SF(\mathcal{A})$ . Note that an arbitrary direct sum of semi-free  $\mathcal{A}^0$ -DG-modules is again semi-free. Hence the triangulated category  $Ho(SF(\mathcal{A}))$  contains arbitrary direct sums. By Theorem 1.11 above it is Karoubian. Hence the category  $Ho(\text{Perf-}\mathcal{A})$  is also Karoubian, and thus it is the Karoubization of the triangulated category  $\mathcal{A}^{tr}$ .

It was pointed to us by Bernhard Keller that the category  $Ho(\text{Perf-}\mathcal{A})$  can be characterized as consisting of *compact* or *small* objects in  $Ho(SF(\mathcal{A})) \simeq D(\mathcal{A})$  [Ke],[Ne],[Ra].

Note that the Yoneda embedding  $h : \mathcal{A} \rightarrow \mathcal{A}^0\text{-mod}$  defines a full and faithful DG functor  $h : \mathcal{A} \rightarrow \text{Perf-}\mathcal{A}$ . Also a DG functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  induces a DG functor  $\text{Ind}_F : \text{Perf-}\mathcal{A} \rightarrow \text{Perf-}\mathcal{B}$  so that the functorial diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{h} & \text{Perf-}\mathcal{A} \\ F \downarrow & & \downarrow \text{Ind}_F \\ \mathcal{B} & \xrightarrow{h} & \text{Perf-}\mathcal{B} \end{array}$$

is commutative.

**Lemma 3.14.** *If  $F : \mathcal{A} \rightarrow \mathcal{B}$  is a quasi-equivalence, then  $\text{Ind}_F : \text{Perf-}\mathcal{A} \rightarrow \text{Perf-}\mathcal{B}$  is also such.*

**Proof.** The quasi-equivalence  $F : \mathcal{A} \rightarrow \mathcal{B}$  induces a quasi-equivalence  $\text{Ind}_F : \alpha_{\mathcal{A}}(\mathcal{A}^{\text{pre-tr}}) \rightarrow \alpha_{\mathcal{B}}(\mathcal{B}^{\text{pre-tr}})$ . Since the subcategories  $Ho(\text{Perf-}\mathcal{A}) \subset Ho(SF(\mathcal{A}))$  and  $Ho(\text{Perf-}\mathcal{B}) \subset Ho(SF(\mathcal{B}))$  are the epaisse envelopes of the subcategories  $Ho(\alpha_{\mathcal{A}}(\mathcal{A}^{\text{pre-tr}}))$  and  $Ho(\alpha_{\mathcal{B}}(\mathcal{B}^{\text{pre-tr}}))$  respectively, the DG functor  $\text{Ind}_F$  induces the quasi-equivalence  $\text{Perf-}\mathcal{A} \rightarrow \text{Perf-}\mathcal{B}$ .  $\square$

**Corollary 3.15.** *If DG categories  $\mathcal{A}$  and  $\mathcal{B}$  are quasi-equivalent, then so are  $\text{Perf-}\mathcal{A}$  and  $\text{Perf-}\mathcal{B}$ .*

**Proof.** This follows from the last lemma.  $\square$

**Lemma 3.16.** *Let  $\mathcal{A}$  be a DG category,  $\mathcal{B} \subset \mathcal{A}$  a full DG subcategory such that  $Ho(\mathcal{B})$  is dense in  $Ho(\mathcal{A})$ , i.e. every object  $A \in Ho(\mathcal{A})$  is a direct summand of an object in  $Ho(\mathcal{B})$ . Then  $Ho(\mathcal{B}^{\text{pre-tr}})$  is dense in  $Ho(\mathcal{A}^{\text{pre-tr}})$ .*

**Proof.** Let  $A, B, C, D \in \mathcal{A}$  and assume that  $A \oplus B$  and  $C \oplus D$  are homotopy equivalent to objects of  $\mathcal{B}$ . Let  $f : A \rightarrow C$  be a closed morphism of degree zero. By part b) of Proposition 3.9 it suffices to show that there exists  $K \in \mathcal{A}^{\text{pre-tr}}$  such that  $\text{Cone}(f) \oplus K$  is homotopy equivalent to an object of  $\mathcal{B}^{\text{pre-tr}}$ . Consider the closed morphism of degree zero  $g : A \oplus B \rightarrow C \oplus D$ , where  $g|_A = f$  and  $g|_B = 0$ . Then

$$\text{Cone}(g) = \text{Cone}(f) \oplus D \oplus B[1],$$

which is homotopy equivalent to an object of  $\mathcal{B}^{\text{pre-tr}}$ , since  $A \oplus B$ , and  $C \oplus D$  are such.  $\square$

**Proposition 3.17.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be DG categories. The natural full and faithful DG functor  $G : \mathcal{A} \otimes \mathcal{B} \rightarrow (\text{Perf-}\mathcal{A}) \otimes \mathcal{B}$  induces a quasi-equivalence  $\text{Ind}_G : \text{Perf-}(\mathcal{A} \otimes \mathcal{B}) \rightarrow \text{Perf-}((\text{Perf-}\mathcal{A}) \otimes \mathcal{B})$ .*



**Proof.** Denote  $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ ,  $\mathcal{D} = (\text{Perf-}\mathcal{A}) \otimes \mathcal{B}$ . Consider the induced full and faithful DG functor

$$\text{Ind}_G : \alpha_{\mathcal{C}}(\mathcal{C}^{\text{pre-tr}}) \rightarrow \alpha_{\mathcal{D}}(\mathcal{D}^{\text{pre-tr}}).$$

Since  $\mathcal{C}^{\text{pre-tr}}$  is DG equivalent to  $(\mathcal{A}^{\text{pre-tr}} \otimes \mathcal{B})^{\text{pre-tr}}$  and  $Ho(\mathcal{A}^{\text{pre-tr}} \otimes \mathcal{B})$  is dense in  $Ho((\text{Perf-}\mathcal{A}) \otimes \mathcal{B})$  it follows from Lemma 3.15 that the category  $Ho(\text{Ind}_G)(Ho(\alpha_{\mathcal{C}}(\mathcal{C}^{\text{pre-tr}})))$  is dense in  $Ho(\alpha_{\mathcal{D}}(\mathcal{D}^{\text{pre-tr}}))$ . Therefore  $Ho(\text{Ind}_G)$  induces an equivalence of Karoubizations  $Ho(\text{Perf-}\mathcal{C})$  and  $Ho(\text{Perf-}\mathcal{D})$  of these categories.  $\square$

**Definition 3.18.** A DG category  $\mathcal{A}$  is *perfect* if it is pretriangulated and the triangulated category  $Ho(\mathcal{A})$  is Karoubian.

Note that a DG category which is quasi-equivalent to a perfect one is by itself perfect.

*Example 3.19.*  $\text{Perf-}\mathcal{A}$  is perfect for any DG category  $\mathcal{A}$ .

**Proposition 3.20.** *If a DG category  $\mathcal{A}$  is perfect, then the (Yoneda) embedding  $\mathcal{A} \hookrightarrow \text{Perf-}\mathcal{A}$  is a quasi-equivalence. In particular  $\text{Perf-}\mathcal{A}$  is quasi-equivalent to  $\text{Perf-}(\text{Perf-}\mathcal{A})$ .*

**Proof.** Similar to the proof of Proposition 3.16  $\square$

**5. Enhanced triangulated categories.** Given a triangulated category  $\mathcal{D}$ , by its *enhancement* we shall mean a pre-triangulated DG category  $\mathcal{A}$  together with an equivalence of triangulated categories  $\epsilon_{\mathcal{A}} : Ho(\mathcal{A}) \rightarrow \mathcal{D}$ . The category  $\mathcal{D}$  is then said to be *enhanced*. Given enhancements  $(\mathcal{A}, \epsilon_{\mathcal{A}})$  and  $(\mathcal{B}, \epsilon_{\mathcal{B}})$  of  $\mathcal{D}$  we call a DG functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  a *quasi-equivalence of enhancements* if  $F$  is a quasi-equivalence and the functors  $\epsilon_{\mathcal{A}}$  and  $Ho(F) \circ \epsilon_{\mathcal{B}}$  are isomorphic. Enhancements  $(\mathcal{A}, \epsilon_{\mathcal{A}})$  and  $(\mathcal{B}, \epsilon_{\mathcal{B}})$  are called quasi-equivalent if there exist enhancements  $(\mathcal{A}_i, \epsilon_{\mathcal{A}_i})$  of  $\mathcal{D}$  and a chain of quasi-equivalences of enhancements

$$\mathcal{A} \leftarrow \mathcal{A}_1 \rightarrow \mathcal{A}_2 \dots \leftarrow \mathcal{A}_n \rightarrow \mathcal{B}.$$

It is natural to consider quasi-equivalence classes of enhancements of a triangulated category.

#### 4. GROTHENDIECK RING OF PRETRIANGULATED CATEGORIES

##### 1. Grothendieck group of pretriangulated categories.

**Definition 4.1.** Denote by  $\mathcal{PT}$  the abelian group generated by quasi-equivalence classes of pretriangulated categories with relations coming

from semi-orthogonal decompositions. Namely, given pretriangulated categories  $\mathcal{A}, \mathcal{B}, \mathcal{C}$ , there is the relation

$$[\mathcal{A}] = [\mathcal{B}] + [\mathcal{C}]$$

in  $\mathcal{PT}$  if there exist pretriangulated categories  $\mathcal{A}', \mathcal{B}', \mathcal{C}'$  which are quasi-equivalent to  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  respectively, such that

- 1)  $\mathcal{B}', \mathcal{C}'$  are DG subcategories of  $\mathcal{A}'$ ,
- 2)  $Ho(\mathcal{B}'), Ho(\mathcal{C}')$  are admissible subcategories of  $Ho(\mathcal{A}')$ , and
- 3)  $Ho(\mathcal{A}') = (Ho(\mathcal{B}'), Ho(\mathcal{C}'))$  is a semiorthogonal decomposition.

We are going to turn the group  $\mathcal{PT}$  into an associative commutative ring by defining the appropriate product of pretriangulated categories.

## 2. $\bullet$ - product of pretriangulated categories.

**Definition 4.2.** Let  $\mathcal{A}_1, \dots, \mathcal{A}_n$  be DG categories. Define the DG category

$$\mathcal{A}_1 \bullet \dots \bullet \mathcal{A}_n := \text{Perf}(\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n).$$

Thus  $\mathcal{A}_1 \bullet \dots \bullet \mathcal{A}_n$  is a perfect DG category and hence the category  $Ho(\mathcal{A}_1 \bullet \dots \bullet \mathcal{A}_n)$  is triangulated and Karoubian.

*Remark 4.3.* The product  $\mathcal{A}_1 \bullet \dots \bullet \mathcal{A}_n$  is functorial in each variable with respect to DG functors.

Proposition 3.14 above implies that  $\bullet$  preserves quasi-equivalence classes of DG categories. Proposition 3.16 implies that the induced operation

$$\bullet : \mathcal{PT} \times \mathcal{PT} \rightarrow \mathcal{PT}$$

is associative. Since the DG categories  $\mathcal{A} \otimes \mathcal{B}$  and  $\mathcal{B} \otimes \mathcal{A}$  are isomorphic, this operation is commutative. We want to show that it is distributive with respect to the addition in the group  $\mathcal{PT}$ .

**Lemma 4.4.** *If a DG functor  $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  is such that  $Ho(F)$  is full and faithful, then the functor  $Ho(\text{Ind}_F) : Ho(\text{Perf-}\mathcal{A}_1) \rightarrow Ho(\text{Perf-}\mathcal{A}_2)$  is also full and faithful.*

**Proof.** Indeed, it follows from part b) of Proposition 3.9 that the exact functor  $Ho(\text{Ind}_F) : \mathcal{A}_1^{tr} \rightarrow \mathcal{A}_2^{tr}$  is full and faithful, hence so is the functor  $Ho(\text{Ind}_F) : Ho(\text{Perf-}\mathcal{A}_1) \rightarrow Ho(\text{Perf-}\mathcal{A}_2)$ .  $\square$

**Lemma 4.5.** *Let  $\mathcal{A}, \mathcal{D}$  be pretriangulated DG categories and  $\mathcal{B} \subset \mathcal{A}$  be a pretriangulated DG subcategory such that  $Ho(\mathcal{B})$  is admissible in  $Ho(\mathcal{A})$ . Then  $\mathcal{B} \bullet \mathcal{D}$  is pretriangulated DG subcategory of  $\mathcal{A} \bullet \mathcal{D}$  such that  $Ho(\mathcal{B} \bullet \mathcal{D})$  is admissible in  $Ho(\mathcal{A} \bullet \mathcal{D})$ .*

**Proof.** Assume for example that  $Ho(\mathcal{B})$  is right admissible in  $Ho(\mathcal{A})$ . There exists a pretriangulated DG subcategory  $\mathcal{C} \subset \mathcal{A}$  such that  $Ho(\mathcal{C}) = Ho(\mathcal{B})^\perp$ . Fix one such subcategory  $\mathcal{C}$ .

Denote by  $i : \mathcal{B} \otimes \mathcal{D} \hookrightarrow \mathcal{A} \otimes \mathcal{D}$  and  $j : \mathcal{C} \otimes \mathcal{D} \hookrightarrow \mathcal{A} \otimes \mathcal{D}$  the embedding DG functors. The functors  $Ho(i)$  and  $Ho(j)$  are full and faithful, so by the last lemma we may identify  $Ho(\mathcal{B} \bullet \mathcal{D})$  and  $Ho(\mathcal{C} \bullet \mathcal{D})$  as full subcategories of  $Ho(\mathcal{A} \bullet \mathcal{D})$ , which we denote  $\tilde{\mathcal{B}}$  and  $\tilde{\mathcal{C}}$  respectively.

Note that  $\tilde{\mathcal{C}} \subset \tilde{\mathcal{B}}^\perp$ . It suffices to prove that for every  $E \in Ho(\mathcal{A} \bullet \mathcal{D})$  there exists an exact triangle

$$E_{\tilde{\mathcal{B}}} \rightarrow E \rightarrow E_{\tilde{\mathcal{C}}},$$

where  $E_{\tilde{\mathcal{B}}} \in \tilde{\mathcal{B}}$  and  $E_{\tilde{\mathcal{C}}} \in \tilde{\mathcal{C}}$ .

Notice that the collection of objects of the subcategory  $Ho(\mathcal{A} \otimes \mathcal{D}) \subset Ho(\mathcal{A} \bullet \mathcal{D})$  classically generates the category  $Ho(\mathcal{A} \bullet \mathcal{D})$ . On the other hand for each object  $A \otimes D \in Ho(\mathcal{A} \otimes \mathcal{D})$  there exists an exact triangle

$$A_{\mathcal{C}} \otimes D \rightarrow A \otimes D \rightarrow A_{\mathcal{B}} \otimes D$$

with  $A_{\mathcal{B}} \in Ho(\mathcal{B})$  and  $A_{\mathcal{C}} \in Ho(\mathcal{C})$ . It remains to apply the Lemma 1.20. The case of  $Ho(\mathcal{B})$  being left admissible is treated similarly.  $\square$

The proof of the last lemma also contains a proof of the next proposition.

**Proposition 4.6.** *Let  $\mathcal{A}, \mathcal{D}$  be pretriangulated DG categories and  $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$  be two DG subcategories such that  $Ho(\mathcal{B})$  and  $Ho(\mathcal{C})$  are admissible in  $Ho(\mathcal{A})$  and  $Ho(\mathcal{A}) = (Ho(\mathcal{B}), Ho(\mathcal{C}))$  is a semi-orthogonal decomposition. Then  $\mathcal{B} \bullet \mathcal{D}, \mathcal{C} \bullet \mathcal{D}$  are DG subcategories in  $\mathcal{A} \bullet \mathcal{D}$ , such that  $Ho(\mathcal{B} \bullet \mathcal{D})$  and  $Ho(\mathcal{C} \bullet \mathcal{D})$  are admissible in  $Ho(\mathcal{A} \bullet \mathcal{D})$  and  $Ho(\mathcal{A} \bullet \mathcal{D}) = (Ho(\mathcal{B} \bullet \mathcal{D}), Ho(\mathcal{C} \bullet \mathcal{D}))$  is a semi-orthogonal decomposition.*

We get the immediate corollary

**Corollary 4.7.** *The operation  $\bullet$  is distributive with respect to addition in the group  $\mathcal{PT}$ . So  $\mathcal{PT}$  is a commutative associative ring.*

Let  $DG(k)^f$  denote the DG category of finite dimensional complexes of  $k$ -vector spaces. Any pretriangulated DG category  $\mathcal{A}$  is quasi-equivalent to  $\mathcal{A} \otimes DG(k)^f$ .

*Remark 4.8.* Denote by  $\mathcal{PT}^+ \subset \mathcal{PT}$  the subgroup generated by quasi-equivalence classes of perfect DG categories. (Note that if  $\mathcal{A}$  is perfect and  $\mathcal{B} \subset \mathcal{A}$  is a pretriangulated subcategory such that  $Ho(\mathcal{B})$  is admissible in  $Ho(\mathcal{A})$ , then  $\mathcal{B}$  is also perfect.) It is clear that  $\mathcal{PT}^+$  is a subring of  $\mathcal{PT}$ . Actually  $\mathcal{PT}^+$  is a unital ring with the unit  $[DG(k)^f]$ .

*Remark 4.9.* Actually one can make  $\mathcal{PT}$  into a ring by using a simpler multiplication  $\mathcal{A}_1 \circ \dots \circ \mathcal{A}_n = (\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n)^{\text{pre-tr}}$ . We chose the operation  $\bullet$  because we like triangulated categories which are Karoubian (it gives them a chance to be saturated).

## 5. A GEOMETRIC EXAMPLE

**1. Standard enhancements of  $D(X)$ .** Let  $X$  be a smooth projective variety. Let us consider the following model for the triangulated category  $D(X)$ . Consider the abelian category  $\mathcal{M}od(\mathcal{O}_X)$  of all (not necessarily quasi-coherent)  $\mathcal{O}_X$ -modules. Put  $D(X) = D_{\text{coh } X}^b(\mathcal{M}od_{\mathcal{O}_X})$ . So the objects of  $D(X)$  and complexes of  $\mathcal{O}_X$ -modules with bounded coherent cohomology.

There are several natural enhancements of the category  $D(X)$ . Consider the pretriangulated DG-category  $C(X)$  consisting of bounded below complexes of  $\mathcal{O}_X$ -modules with bounded coherent cohomology. Let  $I(X)$  denote the full pretriangulated subcategory of  $C(X)$  consisting of complexes of injective  $\mathcal{O}_X$ -modules. Denote the composition of the natural functors  $Ho(I(X)) \rightarrow Ho(C(X)) \rightarrow D(X)$  by  $\epsilon_{I(X)}$ . It is well known that  $\epsilon_{I(X)}$  is an equivalence.

**Definition 5.1.** Let  $(\mathcal{A}, \epsilon_{\mathcal{A}})$  be an enhancement of  $D(X)$  which is quasi-equivalent to  $(I(X), \epsilon_{I(X)})$ . We call  $(\mathcal{A}, \epsilon_{\mathcal{A}})$  a *standard* enhancement of  $D(X)$ .

In fact we believe that all enhancements of  $D(X)$  are standard.

The next lemma is in this direction.

**Lemma 5.2.** *Let  $\mathcal{A} \subset C(X)$  be a full pretriangulated DG subcategory such that the natural functor  $\epsilon_{\mathcal{A}} : Ho(\mathcal{A}) \rightarrow D(X)$  is an equivalence. Then  $(\mathcal{A}, \epsilon_{\mathcal{A}})$  is a standard enhancement of  $D(X)$ .*

**Proof.** Consider the following DG category  $\mathcal{C}$ . Objects of  $\mathcal{C}$  are triples  $(A, I, \gamma)$ , where  $A \in \mathcal{A}$ ,  $I \in I(X)$ , and  $\gamma : A \rightarrow I$  is an injective quasi-isomorphism of complexes. The complex  $\text{Hom}_{\mathcal{C}}((A, I, \gamma), (B, J, \delta))$  is a subcomplex of  $\text{Hom}_{I(X)}(I, J)$  consisting of morphisms which map  $A$  to  $B$ . We have two obvious DG functors

$$\mathcal{A} \xleftarrow{\phi} \mathcal{C} \xrightarrow{\psi} I(X), \quad A \leftarrow (A, I, \gamma) \rightarrow I.$$

*Claim.* The DG functors  $\phi$  and  $\psi$  are quasi-equivalences.

The functor  $\phi$  is surjective on objects since every complex in  $C(X)$  has a (bounded below) injective resolution. The functor  $Ho(\psi)$  is essentially surjective, because  $Ho(\mathcal{A})$  is equivalent to  $D(X)$ . So it remains

to prove that the functors  $Ho(\phi)$  and  $Ho(\psi)$  are full and faithful. Fix  $S = (A, I, \gamma)$ ,  $T = (B, J, \delta)$  and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(S, T) & \xrightarrow{\phi_{S,T}} & \mathrm{Hom}_{\mathcal{A}}(A, B) \\ \psi_{S,T} \downarrow & & \downarrow \delta_* \\ \mathrm{Hom}_{I(X)}(I, J) & \xrightarrow{\gamma^*} & \mathrm{Hom}_{C(X)}(A, J). \end{array}$$

The map  $\gamma^*$  is a quasi-isomorphism, since  $J$  is injective. The map  $\delta_*$  is a quasi-isomorphism, because  $Ho(\mathcal{A})$  is equivalent to  $D(X)$ . So it suffices to prove that  $\phi_{S,T}$  is a quasi-isomorphism.

Since the complex  $J$  is injective the map  $\phi_{S,T}$  is surjective and surjective on cycles. Let  $f \in \mathrm{Hom}_{\mathcal{C}}(S, T)$  be a cycle such that  $\phi_{S,T}f = dg$  for some  $g \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ . It suffices to prove that  $f$  is a boundary in  $\mathrm{Hom}_{\mathcal{C}}(S, T)$ . Choose  $\tilde{g} \in \mathrm{Hom}_{\mathcal{C}}(S, T)$  such that  $\phi_{S,T}\tilde{g} = g$ . Replacing  $f$  by  $f - d\tilde{g}$  we may assume that  $\phi_{S,T}f = 0$ . But then  $f$  is a cycle in  $\mathrm{Hom}_{C(X)}(I/A, J)$ . Since  $I/A$  is acyclic and  $J$  is injective,  $f$  is a boundary in  $\mathrm{Hom}_{C(X)}(I/A, J)$ ; hence also in  $\mathrm{Hom}_{\mathcal{C}}(S, T)$ . This proves the claim.

To prove the lemma it suffices to show that the functors  $\epsilon_{\mathcal{A}} \cdot Ho(\phi)$  and  $\epsilon_{I(X)} \cdot Ho(\psi)$  from  $Ho(\mathcal{C})$  to  $D(X)$  are isomorphic. Fix an object  $(A, I, \gamma) \in \mathcal{C}$ . Then  $\epsilon_{\mathcal{A}} \cdot Ho(\phi)((A, I, \gamma)) = A$ ,  $\epsilon_{I(X)} \cdot Ho(\psi)((A, I, \gamma)) = I$  and the required quasi-isomorphism is the embedding  $\gamma : A \rightarrow I$ .  $\square$

**Definition 5.3.** Fix a full subcategory  $\mathcal{C} \subset D(X)$ . Denote by  $I(\mathcal{C}) \subset I(X)$  the full DG subcategory consisting of complexes, which are quasi-isomorphic to objects in  $\mathcal{C}$ . Clearly, the obvious functor  $\epsilon_{I(\mathcal{C})} : Ho(I(\mathcal{C})) \rightarrow \mathcal{C}$  is an equivalence. An enhancement  $(\mathcal{A}, \epsilon_{\mathcal{A}})$  of  $\mathcal{C}$  is called *standard* if it is quasi-equivalent to  $(I(\mathcal{C}), \epsilon_{I(\mathcal{C})})$ . Notice that this notion depends not only on the category  $\mathcal{C}$ , but also on the given embedding  $\mathcal{C} \subset D(X)$ . Sometimes we will refer to  $c\mathcal{A}$  alone as a standard enhancement of  $\mathcal{C}$ , if it is clear what the functor  $\epsilon_{\mathcal{A}}$  is.

**Lemma 5.4.** *Let  $\mathcal{B} \subset C(X)$  be a full DG subcategory such that the corresponding functor  $\epsilon_{\mathcal{B}} : Ho(\mathcal{B}) \rightarrow D(X)$  is full and faithful. Denote by  $\mathcal{C} \subset D(X)$  the image of the functor  $\epsilon_{\mathcal{B}}$ . Then  $(\mathcal{B}, \epsilon_{\mathcal{B}})$  is a standard enhancement of  $\mathcal{C}$ .*

**Proof.** Same as that of last lemma.  $\square$

The main result of this section is the following theorem.

**Theorem 5.5.** *Let  $X_1, \dots, X_n$  be smooth projective varieties over  $k$ . Then the DG category  $I(X_1) \bullet \dots \bullet I(X_n)$  is quasi-isomorphic to  $I(X_1 \times \dots \times X_n)$ , i.e.*

$$[I(X_1)] \bullet \dots \bullet [I(X_n)] = [I(X_1 \times \dots \times X_n)].$$

Since the operation  $\bullet$  is associative up to quasi-equivalence it suffices to prove the theorem for  $n = 2$ . Put  $X_1 = X$ ,  $X_2 = Y$ .

For a proof of the theorem it will be convenient to use a standard enhancement of  $D(X)$  which we presently define. Let  $\mathcal{P}(X)$  denote the full subcategory of  $D(X)$  consisting of perfect complexes, i.e. finite complexes of vector bundles. It is well known that every object in  $D(X)$  is isomorphic to an object in  $\mathcal{P}(X)$ . Hence the embedding  $\mathcal{P}(X) \subset D(X)$  is an equivalence. Choose a finite affine covering  $\mathcal{U}$  of  $X$ . For any  $P \in \mathcal{P}(X)$  consider its Čech resolution  $P \rightarrow C_{\mathcal{U}}(P)$  defined by  $\mathcal{U}$ . Thus  $C_{\mathcal{U}}(P)$  is a finite complex of quasi-coherent sheaves which are direct sums of sheaves  $P_U^j := i_* i^* P^j$ , where  $i : U \hookrightarrow X$  is the open embedding of an affine open subset  $U$ , which is the intersection of some elements from  $\mathcal{U}$ . Let  $\mathcal{P}(\mathcal{U}) \subset C(X)$  denote the minimal full DG subcategory which contains all complexes  $C_{\mathcal{U}}(P)$  for  $P \in \mathcal{P}(X)$  and is closed under taking cones of closed morphisms of degree zero. Thus  $\mathcal{P}(\mathcal{U})$  is (strongly) pretriangulated. (We could denote the DG category  $\mathcal{P}(\mathcal{U})$  by  $\mathcal{P}^+(\mathcal{U})$  (resp. the complex  $C_{\mathcal{U}}(P)$  by  $C_{\mathcal{U}}^+(P)$ ) as later (proof of Lemma 6.4) we will also consider the "dual" enhancement  $\mathcal{P}^-(\mathcal{U})$  using the left Čech resolutions  $C_{\mathcal{U}}^-(P) \rightarrow P$ .)

We have the obvious functor

$$\epsilon_{\mathcal{P}(\mathcal{U})} : Ho(\mathcal{P}(\mathcal{U})) \rightarrow D(X).$$

**Lemma 5.6.**  *$\epsilon_{\mathcal{P}(\mathcal{U})}$  is an equivalence.*

**Proof.** Since  $\mathcal{P}(X) \hookrightarrow D(X)$  is an equivalence it follows that  $\epsilon_{\mathcal{P}(\mathcal{U})}$  is essentially surjective. It remains to prove that  $\epsilon_{\mathcal{P}(\mathcal{U})}$  is full and faithful. Fix  $P, Q \in \mathcal{P}(X)$ . It suffices to prove that

- 1)  $\text{Hom}_{Ho(C(X))}(P, C_{\mathcal{U}}(Q)) = \text{Hom}_{D(X)}(P, Q)$ ;
- 2)  $\text{Hom}_{Ho(C(X))}(P, C_{\mathcal{U}}(Q)) = \text{Hom}_{Ho(C(X))}(C_{\mathcal{U}}(P), C_{\mathcal{U}}(Q))$ .

By devissage we may assume that complexes  $P$  and  $Q$  are vector bundles placed in degree 0. Let  $i : U \hookrightarrow X$  be an embedding of an affine open subset. Then for any  $n$

$$\text{Hom}_{D(X)}(P, i_* i^* Q[n]) = \text{Hom}_{D(U)}(i^* P, i^* Q[n]) = 0, \quad \text{if } n \neq 0,$$

since  $i^* P$  is a vector bundle on the affine variety  $U$ . This implies 1).

It remains to prove 2). Let  $U, V \subset X$  be open subsets. Note that  $\text{Hom}_{\mathcal{O}_X}(P_U, Q_V) = 0$  if  $V \not\subset U$ . Assume that  $V \subset U$ . Then

$$\text{Hom}_{\mathcal{O}_X}(P_U, Q_V) = \text{Hom}_{\mathcal{O}_V}(P|_V, Q|_V) = \text{Hom}_{\mathcal{O}_V}(\mathcal{O}_X|_V, (P^* \otimes Q)|_V).$$

So we may assume that  $P = \mathcal{O}_X$ . Let  $Q_W$  be one of the summands in the complex  $C_{\mathcal{U}}(Q)$ . (We assume that  $Q_W$  is shifted to degree zero).

Then it follows that the complex  $\mathrm{Hom}_{\mathcal{O}_X}(C_{\mathcal{U}}(\mathcal{O}_X), Q_W)$  is acyclic except in degree zero and

$$H^0(\mathrm{Hom}(C_{\mathcal{U}}(\mathcal{O}_X), Q_W)) = \Gamma(W, Q) = \mathrm{Hom}(\mathcal{O}_X, Q_W).$$

This proves the lemma.  $\square$

It follows from Lemma 5.4 that  $(\mathcal{P}(\mathcal{U}), \epsilon_{\mathcal{P}(\mathcal{U})})$  is a standard enhancement of  $D(X)$ .

Given smooth projective varieties  $X, Y$  choose affine coverings  $\mathcal{U}$  and  $\mathcal{V}$  of  $X$  and  $Y$  respectively. Then  $\mathcal{U} \times \mathcal{V}$  is an affine covering of  $X \times Y$ . Given  $P \in \mathcal{P}(X)$ ,  $Q \in \mathcal{P}(Y)$ ,  $U \in \mathcal{U}$ ,  $V \in \mathcal{V}$  we have  $P_U \boxtimes Q_V = (P \boxtimes Q)_{U \times V}$ . This defines a DG functor

$$\boxtimes : \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}) \rightarrow \mathcal{P}(\mathcal{U} \times \mathcal{V}).$$

**Lemma 5.7.** *The DG functor  $\boxtimes$  is full and faithful.*

**Proof.** Let  $P, P'$  and  $Q, Q'$  be vector bundles on  $X$  and  $Y$  respectively and fix  $U, U' \in \mathcal{U}$ ,  $V, V' \in \mathcal{V}$ . It suffices to prove that the natural map

$$\mathrm{Hom}(P_U, P'_{U'}) \otimes \mathrm{Hom}(Q_V, Q'_{V'}) \rightarrow \mathrm{Hom}((P \boxtimes Q)_{U \times V}, (P' \boxtimes Q')_{U' \times V'})$$

is an isomorphism, where all Hom's are taken in the usual categories of quasi-coherent sheaves. Note that both left and right hand sides are zero if  $U' \not\subset U$  or  $V' \not\subset V$ . So we may assume that  $U' \subset U$  and  $V' \subset V$ . Using the adjunction of direct and inverse image functors the question is reduced to the following "affine" situation: Let  $A$  and  $B$  be noetherian  $k$ -algebras,  $M, M'$  and  $N, N'$  be  $A$ - and  $B$ -modules respectively. Assume that the modules  $M$  and  $N$  are finitely generated. Then the natural map

$$\mathrm{Hom}_A(M, M') \otimes_k \mathrm{Hom}_B(N, N') \rightarrow \mathrm{Hom}_{A \otimes B}(M \otimes N, M' \otimes N')$$

is an isomorphism. By taking resolutions

$$F_1 \rightarrow F_0 \rightarrow M \rightarrow 0, \quad G_1 \rightarrow G_0 \rightarrow N \rightarrow 0,$$

where  $F$ 's and  $G$ 's are free modules of finite rank we may assume that  $M = A$  and  $N = B$ , in which case the assertion is clear.  $\square$

Since the DG functor  $\boxtimes$  is full and faithful the following diagram of DG functors is commutative.

$$\begin{array}{ccc} \mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}) & \xrightarrow{h} & (\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))^0\text{-mod} \\ \boxtimes \downarrow & & \uparrow \mathrm{Res}_{\boxtimes} \\ \mathcal{P}(\mathcal{U} \times \mathcal{V}) & \xrightarrow{h} & \mathcal{P}(\mathcal{U} \times \mathcal{V})^0\text{-mod} \end{array}$$

The DG category  $\mathcal{P}(\mathcal{U} \times \mathcal{V})$  is pretriangulated and perfect (since  $D(X \times Y)$  is Karoubian), so the DG functor  $h : \mathcal{P}(\mathcal{U} \times \mathcal{V}) \rightarrow \mathcal{P}(\mathcal{U} \times \mathcal{V})^0\text{-mod}$  induces a quasi-equivalence  $\mathcal{P}(\mathcal{U} \times \mathcal{V}) \rightarrow \mathrm{Perf}\text{-}\mathcal{P}(\mathcal{U} \times \mathcal{V})$  (3.19).

Let  $\mathcal{A} \subset \mathcal{P}(\mathcal{U} \times \mathcal{V})$  be the smallest full pretriangulated DG subcategory, which contains  $\boxtimes(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))$ . The commutativity of the above diagram implies that the DG functor  $\text{Res}_{\boxtimes} \cdot h$  maps  $\mathcal{A}$  to  $\alpha((\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V}))^{\text{pre-tr}})$  and it induces functor

$$Ho(\text{Res}_{\boxtimes} \cdot h) : Ho(\mathcal{A}) \rightarrow Ho(\text{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})))$$

is full and faithful.

By Theorems 1.9 and 1.10 the subcategory  $\boxtimes(D(X) \times D(Y))$  classically (even strongly) generates  $D(X \times Y)$ . Therefore the triangulated subcategory  $Ho(\mathcal{A}) \subset Ho(\mathcal{P}(\mathcal{U} \times \mathcal{V}))$  is dense, and it follows that  $\text{Res}_{\boxtimes} \cdot h$  maps  $\mathcal{P}(\mathcal{U} \times \mathcal{V})$  to  $\text{Perf}(\mathcal{P}(\mathcal{U}) \otimes \mathcal{P}(\mathcal{V})) = \mathcal{P}(\mathcal{U}) \bullet \mathcal{P}(\mathcal{V})$  and is a quasi-equivalence of these categories. This proves the theorem.

The above proof of Theorem 5.5 gives us a more precise statement: standard enhancements are compatible with products. Namely, we have the following proposition.

**Proposition 5.8.** *Let  $X_1, \dots, X_n$  be smooth projective varieties. For each  $i$  choose a standard enhancement  $(\mathcal{A}_i, \epsilon_i)$  of  $D(X_i)$ . Then there exists an equivalence of triangulated categories*

$$\epsilon : Ho(\mathcal{A}_1 \bullet \dots \bullet \mathcal{A}_n) \rightarrow D(X_1 \times \dots \times X_n),$$

*which makes  $(\mathcal{A}_1 \bullet \dots \bullet \mathcal{A}_n, \epsilon)$  a standard enhancement of  $D(X_1 \times \dots \times X_n)$  and makes the following diagram commutative*

$$\begin{array}{ccc} \times Ho(\mathcal{A}_i) & \subset & Ho(\otimes \mathcal{A}_i) \xrightarrow{Ho(h)} Ho(\mathcal{A}_1 \bullet \dots \bullet \mathcal{A}_n) \\ \downarrow \times \epsilon_i & & \downarrow \epsilon \\ \times D(X_i) & \xrightarrow{\boxtimes} & D(X_1 \times \dots \times X_n). \end{array}$$

**Proof.** Let  $\mathcal{U}_i$  be a finite affine covering of  $X_i$ . The last part of the proof of Theorem 5.5 implies the proposition for standard enhancements  $\mathcal{A}_i = \mathcal{P}(\mathcal{U}_i)$  (the proof there is presented for  $n = 2$ , but the general case is the same). The case of a general standard enhancement now follows from the functoriality of the product  $\bullet$  in each variable (Remark 4.3).  $\square$

## 6. AN APPLICATION: REPRESENTABILITY OF STANDARD FUNCTORS

**Definition 6.1.** Let  $X$  and  $Y$  be smooth projective varieties. A covariant (resp. contravariant) functor  $F : D(X) \rightarrow D(Y)$  is *standard* if there exist standard enhancements  $(\mathcal{A}, \epsilon_{\mathcal{A}})$  and  $(\mathcal{B}, \epsilon_{\mathcal{B}})$  of  $D(X)$  and  $D(Y)$  respectively and a covariant (resp. a contravariant) DG functor



$\tilde{F} : \mathcal{A} \rightarrow \mathcal{B}$  such that the functorial diagram

$$\begin{array}{ccc} Ho(\mathcal{A}) & \xrightarrow{Ho(\tilde{F})} & Ho(\mathcal{B}) \\ \epsilon_{\mathcal{A}} \downarrow & & \downarrow \epsilon_{\mathcal{B}} \\ D(X) & \xrightarrow{F} & D(Y) \end{array}$$

is commutative up to an isomorphism. We call  $\tilde{F}$  a DG *lift* of  $F$ .

Note that a standard functor is necessarily exact.

**Conjecture 6.2.** *All exact functors between  $D(X)$  and  $D(Y)$  are standard.*

*Example 6.3.* The Serre functor  $S = S_X : D(X) \rightarrow D(X)$  is standard. Indeed, the functor  $S_X$  is tensoring by a line bundle  $\omega_X$  and then shifting by the dimension of  $X$ , so it lifts for example to the standard enhancement  $I(X)$ .

Let  $X$  be a smooth projective variety. Recall the anti-involution  $\mathbf{D} : D(X) \rightarrow D(X)$

$$\mathbf{D}(\cdot) = \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\cdot, \mathcal{O}_X).$$

**Lemma 6.4.** *The functor  $\mathbf{D}$  is standard.*

**Proof.** Choose a finite affine covering  $\mathcal{U}$  of  $X$  and consider the full DG subcategory  $\mathcal{P}^-(\mathcal{U}) \subset C(X)$  defined similarly to  $\mathcal{P}(\mathcal{U})$  above except we use left Čech resolutions instead of right ones. Namely, given a perfect complex  $T \in D(X)$  consider its resolution  $C_{\mathcal{U}}^-(T)$ ; it consists of direct sums of  $\mathcal{O}_X$ -modules of the form  $i_! i^* T^j$ , where  $i : U \hookrightarrow X$  is the embedding of an open subset  $U$ , which is the intersection of a few subsets from  $\mathcal{U}$ . As in the case of  $\mathcal{P}(\mathcal{U})$  one proves that the tautological functor  $\epsilon_{\mathcal{P}^-(\mathcal{U})} : Ho(\mathcal{P}^-(\mathcal{U})) \rightarrow D(X)$  is an equivalence, so that  $(\mathcal{P}^-(\mathcal{U}), \epsilon_{\mathcal{P}^-(\mathcal{U})})$  is a standard enhancement of  $D(X)$ . Note that objects of  $\mathcal{P}^-(\mathcal{U})$  are acyclic for the functor  $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\cdot, \mathcal{O}_X)$ . Hence for a perfect complex  $T$  we have

$$\mathbf{D}(T) = \underline{\mathrm{Hom}}(C_{\mathcal{U}}^-(T), \mathcal{O}_X).$$

But notice that the complex  $\underline{\mathrm{Hom}}(C_{\mathcal{U}}^-(T), \mathcal{O}_X)$  is equal to the complex  $C_{\mathcal{U}}(\underline{\mathrm{Hom}}(T, \mathcal{O}_X)) \in \mathcal{P}(\mathcal{U})$ . Thus we have lifted the duality  $\mathbf{D} : D(X) \rightarrow D(X)$  to a contravariant DG functor between the enhancements  $\mathcal{P}^-(\mathcal{U})$  and  $\mathcal{P}(\mathcal{U})$ , which shows that  $\mathbf{D}$  is standard.  $\square$

**Lemma 6.5.** *The collection of standard covariant functors is closed under taking the (left and right) adjoints, inverses (in case the functor is an equivalence) and taking composition of functors.*

**Proof.** Let  $X$  and  $Y$  be smooth projective varieties. Let  $F : D(X) \rightarrow D(Y)$  be an exact functor which is standard. More precisely, assume that there exist standard enhancements  $(\mathcal{A}, \epsilon_{\mathcal{A}})$  and  $(\mathcal{B}, \epsilon_{\mathcal{B}})$  of  $D(X)$  and  $D(Y)$  respectively and that there is DG lift  $\tilde{F} : \mathcal{A} \rightarrow \mathcal{B}$  of the functor  $F$ .

Notice that the DG categories  $\mathcal{A}$  and  $\mathcal{B}$  are perfect, hence the Yoneda DG functors  $h : \mathcal{A} \rightarrow \text{Perf-}\mathcal{A}$ ,  $h : \mathcal{B} \rightarrow \text{Perf-}\mathcal{B}$ , are quasi-equivalences. Thus  $(\text{Perf-}\mathcal{A}, \epsilon_{\mathcal{A}} \cdot Ho(h)^{-1})$  and  $(\text{Perf-}\mathcal{B}, \epsilon_{\mathcal{B}} \cdot Ho(h)^{-1})$  are also standard enhancements of  $D(X)$  and  $D(Y)$  respectively and

$$\text{Ind}_{\tilde{F}} : \text{Perf-}\mathcal{A} \rightarrow \text{Perf-}\mathcal{B}$$

is another DG lift of  $F$ . Its right adjoint DG functor  $\text{Res}_{\tilde{F}}$  is a DG lift of the right adjoint to  $F$ . This proves that the right adjoint to  $F$  and hence the inverse of  $F$  (in case  $F$  is an equivalence) are also standard.

Next we treat the composition of functors. Namely, let  $Z$  be another smooth projective variety and  $G : D(Y) \rightarrow D(Z)$  be a standard functor with a DG lift  $\tilde{G} : \mathcal{C} \rightarrow \mathcal{D}$  for some standard enhancements  $(\mathcal{C}, \epsilon_{\mathcal{C}})$ ,  $(\mathcal{D}, \epsilon_{\mathcal{D}})$  of  $D(Y)$  and  $D(Z)$  respectively. If there exists a quasi-equivalence of enhancements  $\mu : \mathcal{B} \rightarrow \mathcal{C}$ , then the functor  $G \cdot F$  has a DG lift  $\tilde{G} \cdot \mu \cdot \tilde{F}$ . If there exists a quasi-equivalence of enhancements  $\nu : \mathcal{C} \rightarrow \mathcal{B}$ , then the functor  $G \cdot F$  has a DG lift

$$\text{Ind}_{\tilde{G}} \cdot \text{Res}_{\nu} \cdot \text{Ind}_{\tilde{F}} : \text{Perf-}\mathcal{A} \rightarrow \text{Perf-}\mathcal{D}.$$

Since in general there exists a chain of quasi-equivalences of standard enhancements connecting  $\mathcal{B}$  and  $\mathcal{C}$  this shows that the functor  $G \cdot F$  is standard.

It remains to show that a left adjoint to  $F$  is standard. Notice that if  $G$  is the right adjoint to  $F$  (which we proved is standard), then  $S_X^{-1} \cdot G \cdot S_Y$  is the left adjoint, which is therefore also standard.  $\square$

**Definition 6.6.** Let  $X$  and  $Y$  be smooth projective varieties. An exact covariant functor  $F : D(X) \rightarrow D(Y)$  is *represented* by an object  $P \in D(X \times Y)$  if there exists an isomorphism of functors

$$F(\cdot) = \mathbb{R}q_*(p^*(\cdot) \otimes^{\mathbb{L}} P),$$

where  $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$  are the projections. We say that  $F$  is *representable* if it is represented by some object  $P$ .

It is important to know that a given functor is representable, since, in particular, it gives us an algebraic cycle on  $X \times Y$  (the Chern character applied to  $P$ ), and construction of algebraic cycles is always a difficult problem in algebraic geometry. Recall the following important theorem of Orlov [Or2].

**Theorem 6.7.** *Let  $X$  and  $Y$  be smooth algebraic varieties and  $F : D(X) \rightarrow D(Y)$  be a covariant exact functor which is full and faithful. Then  $F$  is representable.*

It is expected that the theorem holds without the assumption on  $F$  being full and faithful.

**Theorem 6.8.** *Let  $X$  and  $Y$  be smooth algebraic varieties and  $F : D(X) \rightarrow D(Y)$  an exact covariant functor which is standard. Then  $F$  is representable.*

We need two lemmas.

**Lemma 6.9.** *In the notation of the theorem consider the (bi-contravariant) functor  $\theta : D(X) \times D(Y) \rightarrow \text{Vect}$  defined as follows:*

$$\theta(A, B) = \text{Hom}_{D(Y)}(B, F(\mathbf{D}A)).$$

*Assume that there exists a contravariant cohomological functor  $\tau : D(X \times Y) \rightarrow \text{Vect}$  such that the functors  $\theta$  and  $\tau \cdot \boxtimes$  are isomorphic. Then  $F$  is representable.*

**Proof.** Since the category  $D(X \times Y)$  is saturated there exists an object  $P \in D(X \times Y)$  and an isomorphism of functors  $\tau(\cdot) = \text{Hom}(\cdot, P)$ . We have the following isomorphisms of functors ([Ha] II.5.15, 5.14, 5.11)

$$\begin{aligned} \text{Hom}_{D(Y)}(B, F(\mathbf{D}A)) &= \theta(A, B) \\ &= \text{Hom}_{D(X \times Y)}(p^*A \otimes^{\mathbb{L}} q^*B, P) \\ &= \text{Hom}_{D(X \times Y)}(q^*B, \mathbb{R}\underline{\text{Hom}}(p^*A, P)) \\ &= \text{Hom}_{D(X \times Y)}(q^*B, \mathbf{D}(p^*A) \otimes^{\mathbb{L}} P) \\ &= \text{Hom}_{D(Y)}(B, \mathbb{R}q_*(\mathbf{D}(p^*A) \otimes^{\mathbb{L}} P)) \end{aligned}$$

By [Ha] II.5.8 there is a functorial isomorphism  $\mathbf{D}(p^*A) = p^*(\mathbf{D}(A))$ . Summarizing we obtain the following functorial isomorphism

$$\text{Hom}_{D(Y)}(B, F(\mathbf{D}A)) = \text{Hom}_{D(Y)}(B, \mathbb{R}q_*(p^*(\mathbf{D}(A)) \otimes^{\mathbb{L}} P)),$$

which implies an isomorphism of functors  $F(\mathbf{D}A) = \mathbb{R}q_*(p^*(\mathbf{D}(A)) \otimes^{\mathbb{L}} P)$ . Since  $\mathbf{D}^2 = Id$  it follows that the functor  $F$  is represented by the object  $P$ .  $\square$

The next lemma explains the role of the assumption on the functor  $F$  to be standard.

**Lemma 6.10.** *A standard functor  $F$  in the above theorem satisfies the assumptions of Lemma 6.9.*

**Proof.** Let  $\tilde{F} : \mathcal{A} \rightarrow \mathcal{B}$  be a DG lift of  $F$  where  $(\mathcal{A}, \epsilon_{\mathcal{A}})$ ,  $(\mathcal{B}, \epsilon_{\mathcal{B}})$  are standard enhancements of  $D(X)$  and  $D(Y)$  respectively. We know that the contravariant functor  $\mathbf{D} : D(X) \rightarrow D(Y)$  has a DG lift  $\tilde{\mathbf{D}} : \mathcal{P}^-(\mathcal{U}) \rightarrow \mathcal{P}(\mathcal{U})$  (Lemma 6.4). First let us show that the composition  $F \cdot \mathbf{D}$  is a standard functor. The argument is similar to the proof of Lemma 6.5 above. Namely, if there exists a quasi-equivalence of enhancements  $\mu : \mathcal{P}(\mathcal{U}) \rightarrow \mathcal{A}$ , then

$$\tilde{F} \cdot \mu \cdot \tilde{\mathbf{D}} : \mathcal{P}^-(\mathcal{U}) \rightarrow \mathcal{B}$$

is a DG lift of  $F \cdot \mathbf{D}$ . If there exists a quasi-equivalence of enhancements  $\nu : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{U})$ , then

$$\text{Ind}_{\tilde{F}} \cdot \text{Res}_{\nu} \cdot h \cdot \tilde{\mathbf{D}} : \mathcal{P}^-(\mathcal{U}) \rightarrow \text{Perf-}\mathcal{B}$$

is a DG lift of  $F \cdot \mathbf{D}$ . In general the standard enhancements  $\mathcal{P}(\mathcal{U})$  and  $\mathcal{A}$  of  $D(X)$  are connected by a chain of quasi-equivalences and we can use the above procedure at each step to show that  $G = F \cdot \mathbf{D}$  is standard. So we may assume that  $G$  has a DG lift  $\tilde{G} : \mathcal{C} \rightarrow \mathcal{D}$  for some standard enhancements  $(\mathcal{C}, \epsilon_{\mathcal{C}})$ ,  $(\mathcal{D}, \epsilon_{\mathcal{D}})$  of  $D(X)$  and  $D(Y)$  respectively.

Consider the DG module  $\tilde{\theta} \in (\mathcal{C} \otimes \mathcal{D})^0\text{-mod}$  defined by

$$\tilde{\theta}(A, B) = \text{Hom}_{\mathcal{D}}(B, \tilde{G}(A)).$$

Note that the functors  $Ho(\tilde{\theta})$  and  $\theta \cdot (\epsilon_{\mathcal{C}} \otimes \epsilon_{\mathcal{D}})$  are isomorphic.

*Claim.* There exists a DG module  $\bar{\theta} \in (\mathcal{C} \bullet \mathcal{D})^0\text{-mod}$  such that  $\bar{\theta} \cdot h = \tilde{\theta}$ , where  $h : \mathcal{C} \otimes \mathcal{D} \rightarrow \mathcal{C} \bullet \mathcal{D}$  is the Yoneda embedding.

Indeed, since  $\mathcal{C} \bullet \mathcal{D}$  is a DG subcategory of  $(\mathcal{C} \otimes \mathcal{D})^0\text{-mod}$  we may define

$$\bar{\theta}(Z) = \text{Hom}_{(\mathcal{C} \otimes \mathcal{D})^0\text{-mod}}(Z, \tilde{\theta}), \quad \text{for } Z \in \mathcal{C} \bullet \mathcal{D}.$$

Notice that if  $Z = h((A, B))$  for  $(A, B) \in \mathcal{C} \otimes \mathcal{D}$ , then

$$\text{Hom}_{(\mathcal{C} \otimes \mathcal{D})^0\text{-mod}}(Z, \tilde{\theta}) = \tilde{\theta}((A, B)),$$

so we have  $\bar{\theta} \cdot h = \tilde{\theta}$ . This proves the claim.

By Proposition 5.9 there exists an equivalence  $\epsilon : Ho(\mathcal{C} \bullet \mathcal{D}) \rightarrow D(X \times Y)$ , such that the diagram

$$\begin{array}{ccccc} Ho(\mathcal{C}) \times Ho(\mathcal{D}) & \subset & Ho(\mathcal{C} \otimes \mathcal{D}) & \xrightarrow{Ho(h)} & Ho(\mathcal{C} \bullet \mathcal{D}) \\ \downarrow \epsilon_{\mathcal{C}} \times \epsilon_{\mathcal{D}} & & & & \downarrow \epsilon \\ D(X) \times D(Y) & & \xrightarrow{\boxtimes} & & D(X \times Y) \end{array}$$

is commutative. Therefore if we put  $\tau = Ho(\bar{\theta}) \cdot \epsilon^{-1}$ , then there exists an isomorphism of functors  $\tau \cdot \boxtimes = \theta$ . This proves the lemma and the theorem.  $\square$

## 7. ANOTHER APPLICATION: A MOTIVIC MEASURE

1. The Grothendieck group  $\Gamma$ .

**Definition 7.1.** Consider the abelian group  $\Gamma = \Gamma_k$  generated by quasi-equivalence classes of pretriangulated categories  $I(X)$  for smooth projective varieties  $X$  over  $k$  with relations coming from semiorthogonal decompositions. Namely, we impose the relation

$$[I(X)] = [I(Y_1)] + \dots + [I(Y_n)],$$

if there exist pretriangulated categories  $\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_n$ , which are quasi-equivalent to  $I(X), I(Y_1), \dots, I(Y_n)$  respectively and the following properties hold

- 1)  $\mathcal{B}_1, \dots, \mathcal{B}_n$  are DG subcategories of  $\mathcal{A}$ ,
- 2)  $Ho(\mathcal{B}_1), \dots, Ho(\mathcal{B}_n)$  are admissible subcategories of  $Ho(\mathcal{A})$ ,
- 3)  $Ho(\mathcal{A}) = (Ho(\mathcal{B}_1), \dots, Ho(\mathcal{B}_n))$  is a semiorthogonal decomposition.

Notice that the group  $\Gamma$  is defined similarly to the group  $\mathcal{PT}$ , except we only use quasi-equivalence classes of pretriangulated  $I(X)$ . In particular there is the obvious group homomorphism  $\beta : \Gamma \rightarrow \mathcal{PT}$  (which may not be injective). Just like in the case of  $\mathcal{PT}$  it follows (using Theorem 5.5) that  $\Gamma$  is a commutative associative ring. Note that  $[I(pt)] = [DG(k)^f]$  is the identity in  $\Gamma$ .

*Remark 7.2.* Semiorthogonal decompositions of derived categories  $D(X)$  for smooth projective  $X$  tend to be rare. Examples are given in [BoOr] and in Propositions 7.3 and 7.5 below. Semiorthogonal summands of categories  $D(X)$  are not always equivalent to categories  $D(Y)$  for a smooth projective  $Y$ . So it might be more natural to include among generators of  $\Gamma$  the (quasi-equivalence classes of) standard enhancements of semiorthogonal summands of categories  $D(X)$ .

**Proposition 7.3.** *Let  $Z$  be a smooth projective variety and  $E$  be a projectivization of a vector bundle of dimension  $d$  on  $Z$ . Then there is the following relation in  $\Gamma$ :*

$$[I(E)] = d[I(Z)].$$

**Proof.** The proof of this and the next proposition is based on two theorems in [Or].

Let  $V \rightarrow Z$  be a vector bundle of dimension  $d$  on  $Z$  such that  $p : E \rightarrow Z$  is its projectivization. There exists a canonical relatively ample line bundle  $\mathcal{O}_V(1)$  on  $E$ . Consider the functors  $p_s^* : D(Z) \rightarrow D(E)$ , where  $p_s^*$  is the composition of the inverse image functor  $p^* = \mathbb{L}p^* : D(Z) \rightarrow D(E)$  with the tensoring by the line bundle  $\mathcal{O}_V(s)$ . The

functor  $p_0^* = p^*$  (and hence also the functors  $p_s^*$  for all  $s$ ) is full and faithful [Or]. Denote by  $D(Z)_s \subset D(E)$  the strictly full subcategory of  $D(E)$  which is the essential image of the functor  $p_s^*$ . It is proved in [Or] that  $(D(Z)_{-d+1}, \dots, D(Z)_0)$  is a semiorthogonal decomposition of  $D(E)$ .

The functor  $p^*$  induces a fully faithful DG functor  $p^* = p_0^* : I(Z) \rightarrow C(E)$ . Consider the commutative diagram of functors

$$\begin{array}{ccc} Ho(I(Z)) & \xrightarrow{Ho(p^*)} & Ho(p^*(I(Z))) \\ \downarrow & & \downarrow \\ D(Z) & \xrightarrow{p^*} & D(E), \end{array}$$

where the vertical arrows are the obvious functors. This shows that the right vertical arrow is full and faithful, because the other three are such. Hence  $p^*(I(Z))$  (which is DG equivalent to  $I(Z)$ ) is a standard enhancement of the category  $D(Z)_0 \subset D(E)$ .

For each  $s$  we may consider the DG functor  $p_s^* : I(Z) \rightarrow C(E)$  which is the composition of  $p^*$  with tensoring by the line bundle  $\mathcal{O}_V(s)$ . Clearly, the image DG category  $p_s^*(I(Z))$  (which is DG equivalent to  $I(Z)$ ) is a standard enhancement of the subcategory  $D(Z)_s \subset D(E)$ . The proposition follows.  $\square$

*Example 7.4.* Taking  $Z = pt$  we deduce that  $[I(\mathbb{P}^{d-1})] = d$  in  $\Gamma$ . This, of course, also follows from the theorem of Beilinson on the resolution of the diagonal in  $\mathbb{P}^n \times \mathbb{P}^n$ .

**Proposition 7.5.** *Let  $Y$  be a smooth projective variety and  $Z \subset Y$  a smooth closed subvariety of codimension  $d$ . Denote by  $X$  the blowup of  $Y$  along  $Z$  with the exceptional divisor  $E \subset X$ . Then there is the following relation in the group  $\Gamma$ :*

$$[I(X)] + [I(Z)] = [I(Y)] + [I(E)].$$

**Proof.** We have the following obvious commutative diagram

$$\begin{array}{ccc} E & \xhookrightarrow{j} & X \\ p \downarrow & & \downarrow \pi \\ Z & \hookrightarrow & Y, \end{array}$$

where  $E$  is the projectivization of a vector bundle of dimension  $d$  over  $Z$ . As in the proof of the last lemma consider the full subcategories  $D(Z)_{-d+1}, \dots, D(Z)_0 \subset D(E)$ . Denote by  $j_{*s}$  the restriction of the direct image functor  $j_* : D(E) \rightarrow D(X)$  to the subcategory  $D(Z)_s$ . The following statements are proved in [Or]:

- 1) the functors  $j_{*s}$  are full and faithful for all  $s$ ;

- 2) the inverse image functor  $\pi^* : D(Y) \rightarrow D(X)$  is full and faithful;  
 3)  $(j_{*(-d+1)}D(Z)_{-d+1}, \dots, j_{*(-1)}D(Z)_{-1}, \pi^*D(Y))$  is a semiorthogonal decomposition of  $D(X)$ .

Hence (as in the proof of the last proposition) it follows that the DG subcategory  $j_{*s}p_s^*(I(Z)) \subset C(X)$  (which is DG equivalent to  $I(Z)$ ) is a standard enhancement of the subcategory  $j_{*s}D(Z)_s \subset D(X)$ . Also  $\pi^*(I(Y))$  (which is DG equivalent to  $I(Y)$ ) is a standard enhancement of  $\pi^*(D(Y))$ . We obtain the following relation in  $\Gamma$

$$[I(X)] = [I(Y)] + (d-1)[I(Z)].$$

It remains to apply the last proposition.  $\square$

**2. A motivic measure.** Let  $\mathcal{V}_k$  denote the collection of  $k$ -varieties. The Grothendieck group  $K_0[\mathcal{V}_k]$  is generated by isomorphism classes of  $k$ -varieties with relations

$$[X] = [Y] + [X - Y],$$

if  $Y \subset X$  is a closed subvariety. The product of varieties over  $k$  makes  $K_0[\mathcal{V}_k]$  a commutative associative ring. Needless to say that it is interesting and important to understand the structure of this ring. However, very little is known. For example, only recently it was proved [Po] that  $K_0[\mathcal{V}_k]$  has zero divisors for any field  $k$ . The following result gives a description of the quotient ring  $K_0[\mathcal{V}_{\mathbb{C}}]/\langle [\mathbb{L}] \rangle$ , where  $\mathbb{L}$  is the class of the affine line  $\mathbb{A}^1$ . Consider the multiplicative monoid  $SB$  of stable birational equivalence classes varieties over  $\mathbb{C}$ . Let  $\mathbb{Z}[SB]$  denote the corresponding monoid ring.

**Theorem 7.6** (LaLu). *There exists a natural isomorphism of rings*

$$K_0[\mathcal{V}_{\mathbb{C}}]/\langle [\mathbb{L}] \rangle \simeq \mathbb{Z}[SB].$$

A ring homomorphism  $K_0[\mathcal{V}_k] \rightarrow A$  is called a *motivic measure*. We claim that there exists a natural (surjective) motivic measure

$$\mu : K_0[\mathcal{V}_{\mathbb{C}}] \rightarrow \Gamma_{\mathbb{C}},$$

whose kernel contains the principal ideal generated by  $\mathbb{L} - 1$  ( $1 = [\text{pt}]$ ).

Indeed, recall the Loojenga-Bittner presentation [Lo],[Bit] of the ring  $K_0[\mathcal{V}_{\mathbb{C}}]$ : it is generated by isomorphism classes of smooth projective varieties with the defining set of relations

$$[X] + [Z] = [Y] + [E],$$

if  $X$  is the blowup of  $Y$  along  $Z$  with the exceptional divisor  $E$ . Then by Proposition 7.5 and Theorem 5.5 the correspondence  $[X] \mapsto [I(X)]$  (for smooth projective  $X$ ) extends to a well defined ring homomorphism  $\mu : K_0[\mathcal{V}_{\mathbb{C}}] \rightarrow \Gamma_{\mathbb{C}}$ . By Example 7.4  $\mu([\mathbb{P}^1]) = 2$ , hence  $\mu(\mathbb{L}) = 1$ .

The induced ring homomorphism

$$\bar{\mu} : K_0[\mathcal{V}_{\mathbb{C}}]/\langle \mathbb{L} - 1 \rangle \rightarrow \Gamma_{\mathbb{C}}$$

is probably not injective (for example, by Mukai's theorem [Mu] the categories  $D(A)$  and  $D(\hat{A})$  are equivalent if  $A$  is an abelian variety and  $\hat{A}$  is its dual), but seems to be close to injective.

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